Chapter 5

Simple Harmonic Motion

Day - 1

INTRODUCTION

Many kinds of motion repeat themselves over and over: the vibration of a quartz crystal in a watch, the swinging pendulum of a grandfather clock, the sound vibrations produced by a clarinet or an organ pipe, and the back-and-forth motion of the pistons in a car engine. This kind of motion, called **periodic moti**on or **oscillation,** is the subject of this chapter. Understanding periodic motion will be essential for our later study of waves, sound, alternating electric currents, and light.

A body that undergoes periodic motion always has a stable equilibrium position. When it is moved away from this position and released, a force or torque comes into play to pull it back toward equilibrium. But by the time it gets there, it has picked up some kinetic energy, so it overshoots, stopping somewhere on the other side, and is again pulled back toward equilibrium. Picture a ball rolling back and forth in a round bowl or a pendulum that swings back and forth past its straight-down position.

We will concentrate on two simple examples of systems that can undergo periodic motions: spring-mass systems and pendulums. We will also study why oscillations often tend to die out with time and why some oscillations can build up to greater and greater displacements from equilibrium when periodically varying forces act.

SIMPLE HARMONIC MOTION

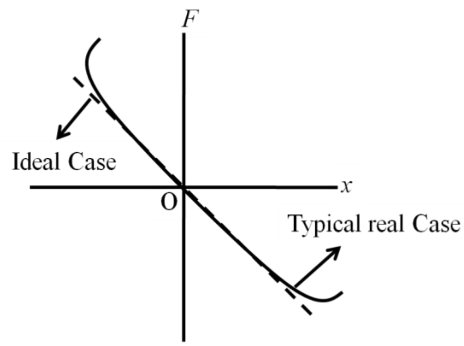
The simplest kind of oscillation occurs when the restoring force F, is directly proportional to the displacement from equilibrium x. This happens if the spring in figure is an ideal one that obeys Hooke's law. The constant of proportionality between F and x is the force constant k. (You may want to review Hooke's law and the definition of the force constant. On either side of the equilibrium position, F, and x always have opposite signs. In Section 6.3 we represented the force acting on a stretched ideal spring as F, = kx. The x-component of force the spring exerts on the body is the negative of this, so the x-component of force F on the body is

$$F = -kx \quad \text{(restoring force exerted by an ideal spring)}$$

This equation gives the correct magnitude and sign of the force, whether x is positive, negative, or zero. The force constant k is always positive and has units of N/m (a useful alternative set of units is kg/s^2). We are assuming that there is no friction, gives the net force on the body. When the restoring force is directly proportional to the displacement from equilibrium, as given by the oscillation is called **simple harmonic motion**, abbreviated **SHM.** The acceleration $a = d^2x/dt^2$ = F/m of a body in SHM is given by

$$a = \frac{d^2x}{dt^2} = -\frac{k}{m}x \quad \text{(Simple Harmonic Motion)}$$

The minus sign means the acceleration and displacement always have opposite signs. This `acceleration is not constant, so don't even think of using the constant- acceleration equations. We'll see shortly how to solve this equation to find the displacement *x* as a function of time. A body that undergoes simple harmonic motion is called a **harmonic oscillator.**

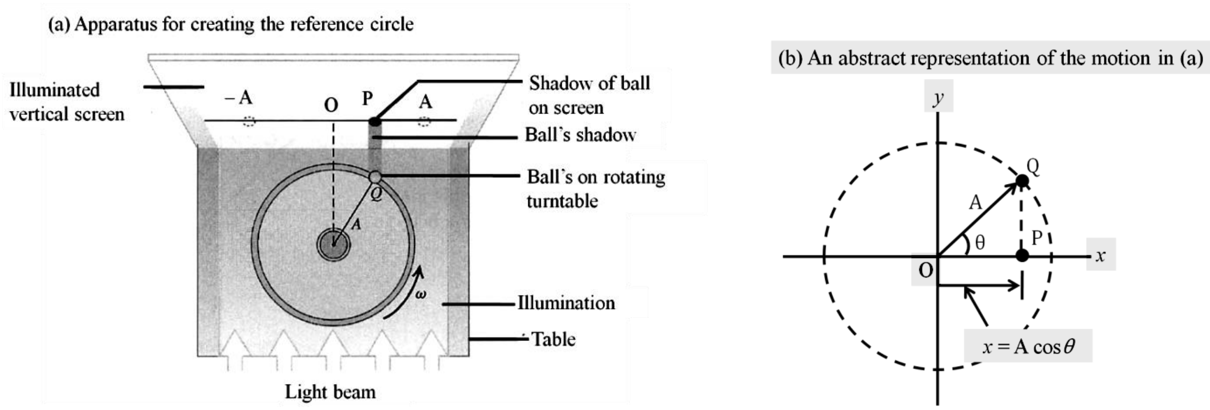


Why is simple harmonic motion important? Keep in mind that not all periodic motions are simple harmonic; in periodic motion in general, the restoring force depends on displacement in a more complicated way than. But in many systems the restoring force is approximately proportional to displacement if the displacement is sufficiently small. That is, if the amplitude is small enough, the oscillations of such systems are approximately simple harmonic and therefore approximately described by. Thus we can use SHM as an approximate model for many different periodic motions, such as the vibration of the quartz crystal in a watch, the motion of a tuning fork, the electric current in an alternating-current circuit, and the oscillations of atoms in molecules and solids.

EQUATIONS OF SIMPLE HARMONIC MOTION

To explore the properties of simple harmonic motion, we must express the displacement *x* of the oscillating body as a function of time, $x(t)$. The second derivative of this function, d^2x/dt^2, must be equal to $(-k/m)$ times the function itself, as required. As we mentioned, the formulas for constant acceleration because the acceleration changes constantly as the displacement x changes. Instead, we'll find $x(t)$ by noticing a striking similarity between SHM and another form of motion that we've already studied in detail.

Figure shows a top view of a horizontal disk of radius A with a ball attached to its rim at point Q. The disk rotates with constant angular speed ω (measured in rad/s), so the ball moves in uniform circular motion. A horizontal light beam shines on the rotating disk and casts a shadow of the ball on a screen. The shadow at point P oscillates back and forth as the ball moves in a circle. We then arrange a body attached to an ideal spring, like the combination shown in figures. So that the body oscillates parallel to the shadow. We will prove that the motion of the body and the motion of the ball's shadow are identical if the amplitude of the body's oscillation is equal to the disk radius A, and if the angular frequency $27\pi f$ of the oscillating body is equal to the angular speed w of the rotating disk. That is, simple harmonic motion is the projection of uniform circular motion onto a diameter.



We can verify this remarkable statement by finding the acceleration of the shadow at P and comparing it to the acceleration of a body undergoing SHM. The circle in which the ball moves so that its projection matches the motion of the oscillating body is called the **reference circle;** we will call the point Q. The reference point. We take the reference circle to lie in the *xy*-plane, with the origin 0 at the center of the circle. At time t the vector OQ from the origin to the reference point Q makes an angle $\theta$the positive *x*-axis. As the point Q moves around the reference circle with constant angular speed ω, the vector OQ rotates with the same angular speed. Such a rotating vector is called a **phasor.** (This term was in use long before the invention of the Star Trek stun gun with a similar name. The phasor method for analyzing oscillations is useful in many areas of physics.

The x-component of the phasor at time t is just the x-coordinate of the point Q:

$$x = A\cos\theta$$

This is also the x-coordinate of the shadow P, which is the projection of Q onto the x-axis. Hence the *x*-velocity of the shadow P along the *x*-axis is equal to the *x*-component of the velocity vector of the reference point Q, and the x-acceleration of P is equal to the *x*-component of the acceleration vector of Q. Since point Q is in uniform circular motion, its acceleration vector $\underset{a_Q}{\rightarrow}$ is l ways directed toward O. Furthermore, the magnitude of $\overrightarrow{aQ}$ is constant and given by the angular speed squared times the radius of the circle.

$$a_Q = \omega^2 A$$

$$a = -a_Q\cos\theta = -\omega^2 A\cos\theta.$$

$$a = -\omega^2 x$$

The acceleration of the point P is directly proportional to the displacement x and always has the opposite sign. These are precisely the hallmarks of simple harmonic motion. Exactly for the acceleration of a harmonic oscillator, provided that the angular speed ω of the reference point Q is related to the force constant k and mass *m* of the oscillating body by

$$\omega^2 = \frac{k}{m}, \quad \text{or} \quad \omega = \sqrt{\frac{k}{m}}$$

We have been using the same symbol ω for the angular speed of the reference point Q and the angular frequency of the oscillating point P. The reason is that these quantities are equal. If point Q makes one complete revolution in time T, then point P goes through one complete cycle of oscillation in the same time; hence T is the period of the oscillation. During time T the point Q moves through 2π radians, so its angular speed is $\omega = 2\pi/T.$ But this is just for the angular frequency of the point P, which verifies our statement about the two interpretations of w. This is why we introduced angular frequency in this quantity makes the connection between oscillation and circular motion. So we reinterpret an expression for the angular frequency of simple harmonic motion for a body of mass *m*, acted on by a restoring force with force constant k:

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(Simple harmonic motion)}$$

When you start a body oscillating in SHM, the value of w is not yours to choose; it is predetermined by the values of *k* and *m*. The units of k are N/m or kg/s^2, so k/m is in (kg/s^2) /kg = S^{-2}. When we take the square root we get S^{-1}, or more properly rad/s because this is an angular frequency (recall that a radian is not a true unit).

The frequency f and period T are –

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{(Simple harmonic motion)}$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{k}{m}} \quad \text{(Simple harmonic motion)}$$

We see from Eq. that a larger mass *m*, with its greater inertia, will have less acceleration, move more slowly, and take a longer time for a complete cycle. In contrast, a stiffer spring (one with a larger force constant k) exerts a greater force at a given deformation x, causing greater acceleration, higher speeds, and a shorter time T per cycle.

Equations show that the period and frequency of simple harmonic motion are completely determined by the mass *m*. and the force constant *k*. In simple harmonic motion. the period and frequency do not depend on the amplitude A. For given values of m and k, the time of one complete oscillation is the same whether the amplitude is large or small. Equation shows why we should expect this. Larger A means that the body reaches larger values of |*x*| and is subjected to larger restoring forces. This increases the average speed of the body over a complete cycle; this exactly compensates for having to travel a larger distance, so the same total 'time is involved. The oscillations of a tuning fork are essentially simple harmonic motion, which means that it always vibrates with the same frequency, independent of amplitude. This is why a tuning fork can be used as a standard for musical pitch. If it were not for this characteristic of simple harmonic motion, it would be impossible to make familiar types of mechanical and electronic clocks run accurately or to play most musical instruments in tune. If you encounter an oscillating body with a period that does depend on the amplitude, the oscillation is not simple harmonic motion.

THE SIMPLE PENDULUM

A simple pendulum is an idealized model consisting of a point mass suspended by a mass less, unstretchable string. When the point mass is pulled to one side of its straight-down equilibrium position and released, it oscillates about the equilibrium position. Familiar situations such as a wrecking ball on a crane's cable or a person on a swing can be modeled as simple pendulums. The path of the point mass (sometimes called a pendulum bob) is not a straight line but the arc of a circle with radius L equal to the length of the string. We use as our coordinate the distance x measured along the arc. If the motion is simple harmonic, the restoring force must be directly proportional to x or (because $x = L\theta$) to θ Is it?

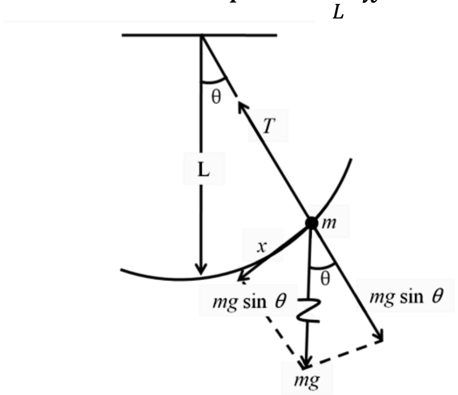
We represent the forces on the mass in terms of tangential and radial components. The restoring force F is the tangential component of the net force:

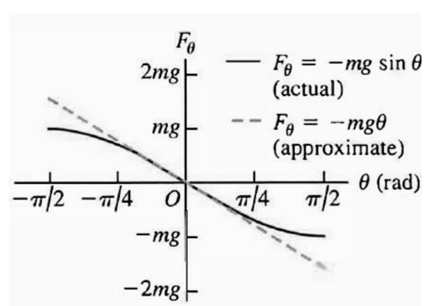
$$F = -mg\sin\theta$$

The restoring force is provided by gravity; the tension T merely acts to make the point mass move in an arc. The restoring force is proportional not to θ but to sine, so the motion is not simple harmonic. However, if the angle e is small, sine is very nearly equal to e in radians. For example, when θ = 0.1 rad (about 6°), sin θ = 0.0998, a difference of only 0.2%. With this approximation,

$$F = -mg\theta = -mg\frac{x}{L}, \quad \text{or}$$

$$F = -\frac{mg}{L}x$$





The restoring force is then proportional to the coordinate for small displacements, and the force constant is $k = mg$/L. The angular frequency ω of a simple pendulum with small amplitude is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}} \quad \text{(Simple pendulum, small amplitude)}$$

The corresponding frequency and period relations are

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \qquad \text{(Simple pendulum, small amplitude)}$$

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}} \qquad \text{(Simple pendulum, small amplitude)}$$

Note that these expressions do not involve the mass of the particle. This is because the restoring force, a component of the particle's weight, is proportional to m. Thus the mass appears on both sides of $\sum \underset{F}{\rightarrow} = m \underset{a}{\rightarrow}$ and cancels out. (This is the same physics that explains why bodies of different masses fall with the same acceleration in a vacuum.) For small oscillations, the period of a pendulum for a given value of g is determined entirely by its length. The dependence on L and g through is just what we should expect. A long pendulum has a longer period than a shorter one. Increasing g increases the restoring force, causing the frequency to increase and the period to decrease. We emphasize again that the motion of a pendulum is only approximately simple harmonic. W hen the amplitude is not small, the departures from simple harmonic motion can be substantial. But how small is "small"? The period can be expressed by an infinite series; when the maximum angular displacement is e, the period T is given by

$$T = 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{1^2}{2^2}\sin^2\frac{\Theta}{2} + \frac{1^2.3^2}{2^2.4^2}\sin^4\frac{\Theta}{2} + \cdots\right)$$

We can compute the period to any desired degree of precision by taking enough terms in the series. We invite you to check that when θ = 15° (on either side of the central position), the true period is longer than that given by the approximate less than 0.5%. The usefulness of the pendulum as a timekeeper depends on the period being very nearly independent of amplitude, provided that the amplitude is small. Thus, as a pendulum clock runs down and the amplitude of the swings decreases a little, the clock still keeps very nearly correct time.

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