Chapter 4

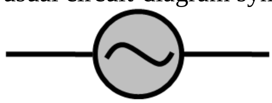
Alternating Current

Day – 1

1. Phasors and Alternating Currents

To supply an alternating current to a circuit, a source of alternating emf or voltage is required. An example of such a source is a coil of wire rotating with constant angular velocity in a magnetic field. This develops a sinusoidal alternating emf and is the prototype of the commercial alternating-current generator or alternator.

We will use the term ac source for any device that supplies a sinusoidal varying voltage (potential difference) υ or current *i*. The usual circuit-diagram symbol for an ac source is



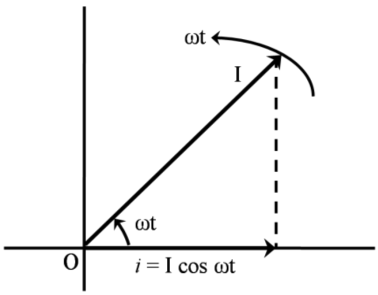
A sinusoidal voltage might be described by a function such as

$$\upsilon = V\cos\omega t$$

In this expression, υ (lowercase) is the instantaneous potential difference; V (uppercase) is the maximum potential difference, which we call the ***voltage amplitude***; and ω is the angular frequency, equal to 2π times the frequency f. In the United States and Canada, commercial electric-power distribution systems always use a frequency of f = 60 Hz, corresponding to $\omega = (2\pi\ \text{rad})(60\ \text{s}^{-1}) = 377$ rad/s; in much of the rest of the world , f = 50 Hz (ω = 314 rad/s) is used. Similarly, a sinusoidal current might be described as

$$i = I\cos\omega t,$$

where *i* (lowercase) is the instantaneous current and I (uppercase) is the maximum current or ***current amplitude.***



A phasor is not a real physical quantity with a direction in space, such as velocity, momentum, or electric field. Rather, it is a geometric entity that helps us to describe and analyze physical quantities that vary sinusoidally with time. In this chapter we will use phasors to add sinusoidal voltages and currents. Combining sinusoidal quantities with phase differences then becomes a matter of vector addition.

The ***rectified average current*** I_{rav} is defined so that during any whole number of cycles, the total charge that flows is the same as though the current were constant with a value equal to I_{rav}. The notation I_{rav} and the name rectified average current emphasize that this is not the average of the original sinusoidal current. In figure the total charge that flows in time t corresponds to the area under the curve of I versus t (recall that I = dq/dt, so q is the integral of t); this area must equal the rectangular area with height I_{rav}. We see that I_{rav} is less than the maximum current I; the two are related by

$$I_{rav} = \frac{2}{\pi} I = 0.6371 \qquad \text{(rectified average value f a sinusoidal current).}$$

A more useful way to describe a quantity that can be either positive or negative is the root-mean-square (rms) value. We used rms values in connection with the speed of molecules in a gas. We square the instantaneous current I, take the average (mean) value of i^2, and finally take the square root of that average. This procedure defines the ***root-mean-square current,*** denoted as I_{rms}. Even when I is negative, i^2 is always positive, so I_{rms} is never zero (unless I is zero at every instant).

Here's how we obtain I_{rms}. If the instantaneous current is given by $i - I \cos \omega t$, then

$$i^2 = I^2 \cos^2 \omega t$$

Using a double-angle formula from trigonometry,

$$\cos^2 A = \frac{1}{2}(1 + \cos 2A),$$

we find

$$i^2 = I^2 \frac{1}{2}(1 + \cos 2\omega t) = \frac{1}{2} I^2 + \frac{1}{2} I^2 \cos 2\omega t.$$

The average of $\cos 2\omega t$ is zero because it is positive half the time and negative half the time. Thus the average of i^2 is simply $I^2/2$. The square root of this is I_{rms}:

$$I_{rms} = \frac{1}{\sqrt{2}} \qquad \text{(root-mean-square value of a sinusoidal current).}$$

In the same way, the root-mean-square value of a sinusoidal voltage with amplitude (maximum value) V is

$$V_{rms} = \frac{V}{\sqrt{2}} \qquad \text{(root-mean-square value of a sinusoidal voltage).}$$

We can convert a rectifying ammeter into a voltmeter by adding a series resistor, just as for the dc case. Meters used for ac voltage and current measurements are nearly always calibrated to read rms values, not maximum or rectified average. Voltages and currents in power distribution systems are always described in terms of their rms values. The usual household power supply, "120-volt ac," has an rms voltage of 120 V. The voltage amplitude is

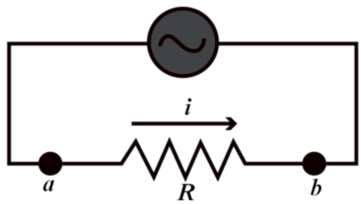
$$V = \sqrt{2}\, V_{rms} = \sqrt{2}\,(120\ V) = 170\ V.$$

1.1 Resistance and Reactance

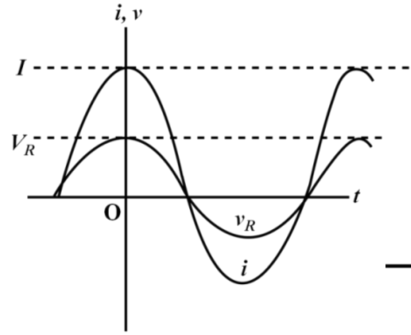
Resistor in an AC circuit

First let's consider a resistor with resistance R through which there is a sinusoidal current given by $i = I \cos \omega t$. The positive direction of current is counter-clockwise around the circuit, as in figure. The current amplitude (maximum current) is I. From Ohm's law the instantaneous potential υ_R of point a with respect to point b (that is, the instantaneous voltage across the resistor) is

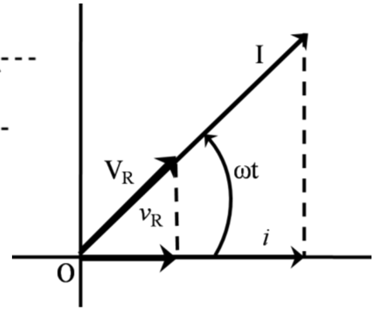
$$\upsilon_R = iR = (IR) \cos \omega t.$$



(a)



(b)



(c)

The maximum voltage V_R, the voltage amplitude, is the coefficient of the cosine function:

$$V_R = IR \qquad \text{(Amplitude of voltage across a resistor, ac circuit).}$$

Hence we can also write

$$\upsilon_R = V_R \cos \omega t.$$

The current I and voltage υ_R are both proportional to $\cos \omega t$, so the current is in phase with the voltage. Equation shows that the current and voltage amplitudes are related in the same way as in a dc circuit. Figure shows graphs of I and υ_R as function of time. The vertical scales for current and voltage are different, so the relative heights of the two curves are not significant. The corresponding phasor diagram is given in figure. Because I and υ_R are in phase and have the same frequency, the current and voltage phasors rotate together; they are parallel at each instant. Their projections on the horizontal axis represent the instantaneous current and voltage, respectively.

Inductor in an Ac circuit

Next, we replace the resistor in figure with a pure inductor with self-inductance L and zero resistance. Again we assume that the current is $i = I \cos \omega t$, with the positive direction of current taken as counterclockwise around the circuit.

lthough there is no resistance, there is a potential difference υ_L between the inductor terminals *a* and b because the current varies with time, giving rise to a self-induced emf. The induced emf in the direction of I is given by equation, $\varepsilon = -L\, di/dt$; however, the voltage υ_L is not simply equal to ε. So we have

$$\upsilon_L = L\frac{di}{dt} = L\frac{d}{dt}(I\cos \omega t) = -I\omega L \sin \omega t.$$

We can also obtain this phase relationship by rewriting equation using the identity

$$\cos(A + 90^0) = -\sin A$$

$$\upsilon_L = I\omega L \cos(\omega t + 90^0).$$

This result shows that the voltage can be viewed as a cosine function with a “head start” of 90^0 relative to the current.

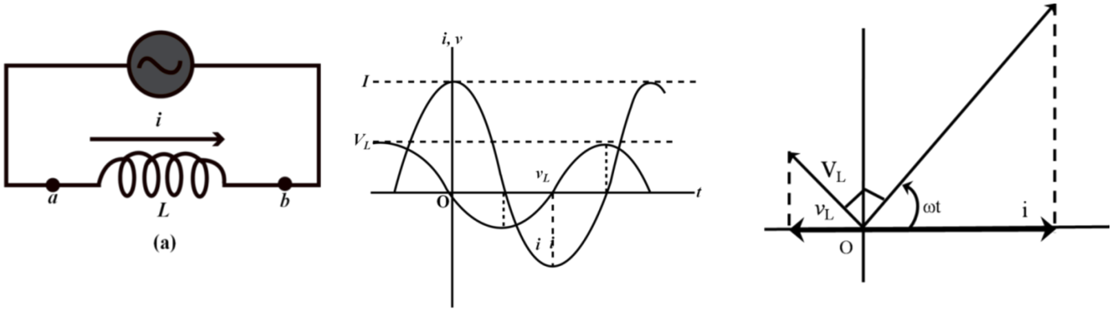
For consistency in later discussions we will usually describe the phase of the voltage relative to the current, not the reverse. Thus if the current *i* in a circuit is

$$i = I\cos \omega t$$

and the voltage υ of one point with respect to another is

$$\upsilon = V\cos(\omega t + \phi),$$

we call ϕ the ***phase angle***; it gives the phase of the voltage relative to the current. For a pure resistor, $\phi = 0$, and for a pure inductor, $\phi = 90^0$.



(a) (b) (c)

From the last equations the amplitude V_L of the inductor voltage is

$$V_L = I\omega L.$$

We define the ***inductive reactance*** X_L of an inductor as

$$X_L = \omega L. \qquad \text{(inductive reactance).}$$

Using X_L, we can write equation $V_L = I\omega L$ in a form similar to equation $V_R = IR$ for a resistor:

$$V_L = IX_L \qquad \text{(amplitude of voltage across an inductor, ac circuit).}$$

Because X_L is the ratio of a voltage and a current, its SI unit is one ohm, the same as for resistance.

Capacitor in an AC circuit

Finally, we connect a capacitor with capacitance C to the source, as in figure, producing a current $i = I\cos\omega t$ through the capacitor. Again, the positive direction of current is counterclockwise around the circuit.

To find the instantaneous voltage v_C across the capacitor, that is, the potential of point a with respect to point b, we first let q denote the charge on the left-hand plate of the capacitor in figure (so–q is the charge on the right-hand plate). The current i is related to q by $i = dq/dt$; with this definition, positive current corresponds to an increasing charge on the left-hand capacitor plate. Then

$$i = \frac{dq}{dt} = I\cos\omega t.$$

Integrating this, we get

$$q = \frac{I}{\omega}\sin\omega t.$$

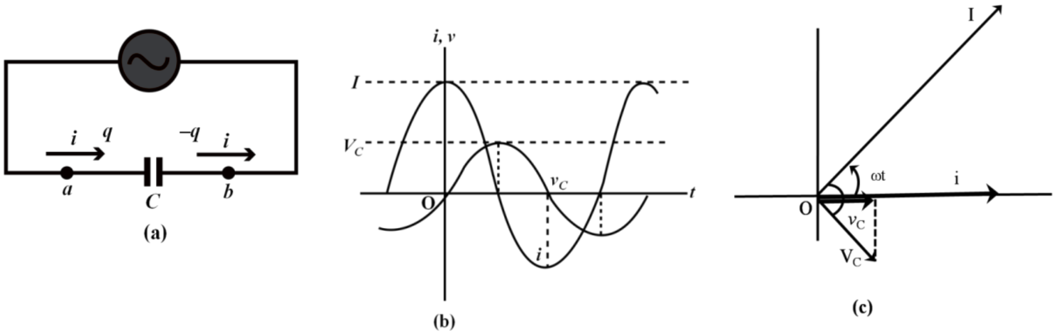
We know that the charge q equals the voltage v_C multiplied by the capacitance, q = C v_C. Using this in the last equation, we find

$$v_C = \frac{I}{\omega C}\sin\omega t.$$

The instantaneous current i is equal to the rate of change dq/dt of the capacitor charge q; since q = Cv_C, i is also proportional to the rate of change of voltage. (Compare to an inductor, for which the situation is reversed and v_L is proportional to the rate of change of i.) Figure shows v_C and i as functions of t. Because $i = dq/dt = C dv_C/dt$, the current has its greatest magnitude when the v_C curve is rising or falling most steeply and is zero when the v_C curve instantaneously levels off at its maximum and minimum values.

The capacitor voltage and current are out of phase by a quarter-cycle. The peaks of voltage occur a quarter-cycle after the corresponding current peaks, and we say that the voltage lags the current by

90^0. The phasor diagram in figure shows this relationship; the voltage phasor is behind the current phasor by a quarter-cycle, or 90^0.



We can also derive this phase difference by rewriting the last equation, using the identity $\cos(A - 90^0) = \sin A$:

$$\upsilon_C = \frac{I}{\omega C}\cos(\omega t - 90^0).$$

This corresponds to a phase angle $\phi = -90^0$. This cosine function has a "late start" of 90^0 compared with the current $i = I\cos\omega t$.

Last two equations show that the maximum voltage V_C (the voltage amplitude) is

$$V_C = \frac{I}{\omega C}.$$

To put this expression in a form similar to equation for a resistor, $V_R = IR$, we define a quantity X_C, called the ***capacitive reactance*** of the capacitor, as

$$X_L = \omega L \qquad \text{(Inductive reactance).}$$

Then

$$V_L = IX_L \qquad \text{(Amplitude of voltage across an inductor, ac circuit).}$$

The SI unit of XC is one ohm, the same as for resistance and inductive reactance, because XC is the ratio of a voltage and a current.

The capacitive reactance of a capacitor is inversely proportional both to the capacitance C and to the angular frequency ω; the greater the capacitance and the higher the frequency, the smaller is the capacitive reactance XC. Capacitors tend to pass high-frequency current and to block low-frequency currents and dc, just the opposite of inductors. A device that preferentially passes signals of high frequency is called a high-pass filter.

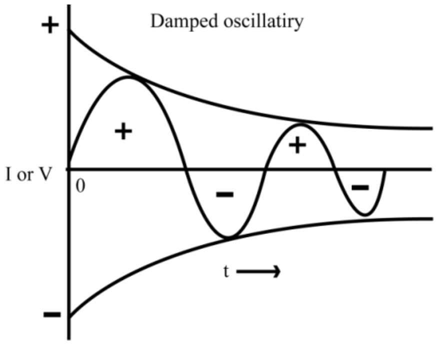
1.2 Alternating Current and Voltage

A time varying current or voltage according to its law of variation may be periodic or non-periodic. In case of periodic current or voltage, the current or voltage is said to be alternating if:

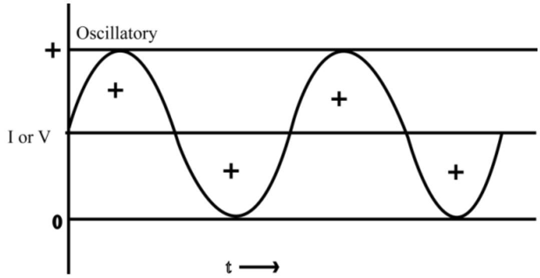
(a) Its amplitude is constant, and

(b) Alternate half cycle is positive and half negative.

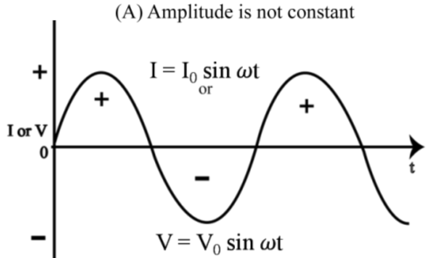
This all is illustrated in figures.



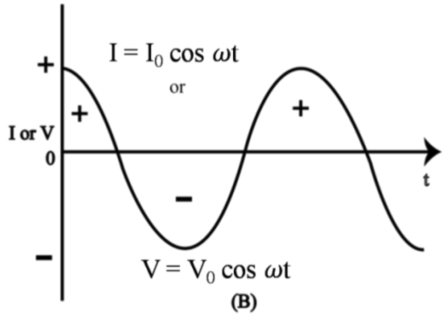
(A) Amplitude is not constant



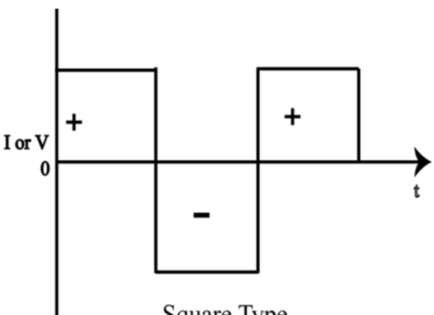
(B) Direction is not changing



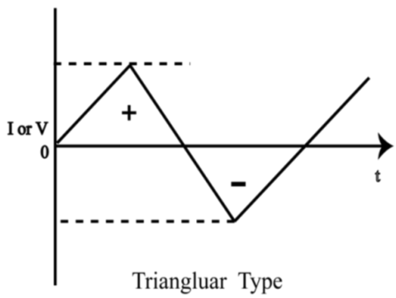
(A)



(B)



Square Type
(A)



Triangluar Type
(B)

(1) It is produced by dynamo or electronic oscillator and is represented by the symbol  .

(2) The frequency of ac in India is 50Hz, *i.e.*,

$$F = 50\text{Hz so } \omega = 2\pi f = 100\pi \text{ rad/s}$$

(3) The ac can be converted into do with the help of rectifier while do into ac with the help of inverter.

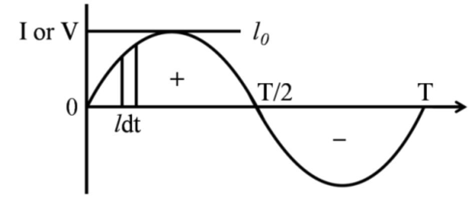
(4) It cannot produce ‘chemical effects of current’ such as electroplating or electrolysis as due to large inertia ions cannot follow the frequency of ac.

(5) The maximum value of alternating current or voltage is called peak value and is represented by I_0 and V_0 respectively. So that alternating current or voltage at any time t in a circuit will be given by,

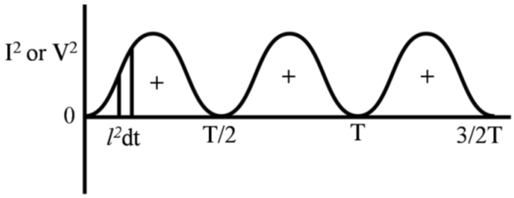
$$I = I_0 \sin \omega t \quad \text{or} \quad V = V_0 \sin \omega t$$

(6) If the average or mean value of alternating current or voltage is defined for full cycle, it will be zero as $\int_0^T \sin \omega t \ dt$ or $\int_0^T \cos \omega t \ dt = 0$ so it defined for positive (or negative) half cycle as,

$$I_{av} \text{ or } I_{mean} = \frac{\int_0^{\frac{T}{2}} I\,dt}{\int_0^{\frac{T}{2}} dt} = \frac{\int_0^{\frac{\pi}{\omega}} I_0 \sin \omega t dt}{\int_0^{\frac{\pi}{\omega}} dt} = \frac{2}{\pi} I_0$$



$$I_{av} = \frac{2}{\pi} I_0$$



$$I_{rms} = \frac{I_0}{\sqrt{2}}$$

(7) Effective, virtual or rms value of alternating current is defined as the square root of the average of I^2 during a complete cycle, i.e.,

$$I_{rms} = \left[\frac{\int_0^T I^2 dt}{\int_0^T dt}\right]^{1/2} = \left[\frac{\int_0^{\frac{2\pi}{\omega}} I_0^2 \sin^2 \omega t dt}{\int_0^{\frac{2\pi}{\omega}} dt}\right]^{1/2}$$

$$I_{rms} = \frac{I_0}{\sqrt{2}} \qquad \left[\text{as } \int_0^T \sin \omega t \; dt = \frac{T}{2}\right]$$

All ac instruments read this value, e.g., if we speak about 220 V alternating voltage we mean V_{rms} = 200V.

(8) As $V_{av} = \frac{2}{\pi} V_0$ and $V_{rms} = \frac{V_0}{\sqrt{2}}$

$V_0 > V_{rm}s > V_{av}$, and V0 = 1.41 V_{rms}

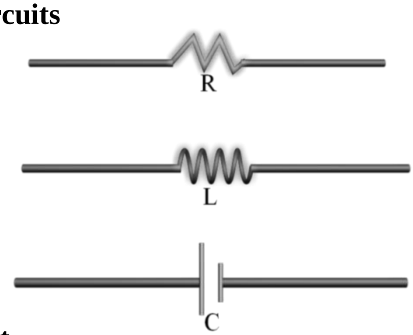
So, if Vrms = 200V, $V_0 = \sqrt{2} \times 220 = 311V$

and $V_{av} = 0.9 \times 220 = 198V$

(9) From factor = I_{rms}/I_{av}

So, for sinusoidal ac $= \frac{I_0}{\sqrt{2}} \times \frac{\pi}{2I_0} = \frac{\pi}{2\sqrt{2}}$

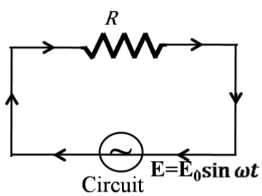
1.3 Circuit Elements in ac Circuits



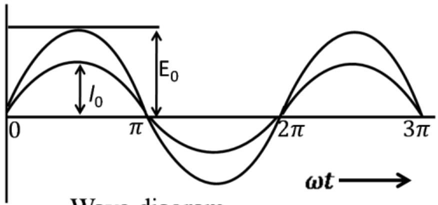
(A) A Resistor in an ac Circuit

$$E = IR, \quad i.e., \quad I = \frac{E}{R}$$

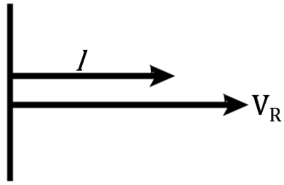
$$I = \frac{E_0}{R} \sin \omega t$$



$$I = I_0 \sin \omega t \quad \text{with } I_0 = \frac{E_0}{R}$$



Wave-diagram



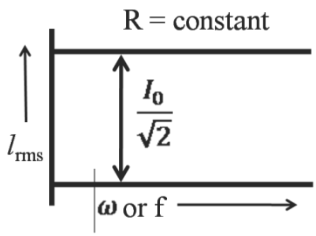
Phasor-diagram

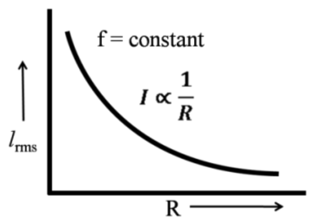
From this it clear that:

(i) The frequency of current in the circuit is ω and is same as that of the applied voltage.

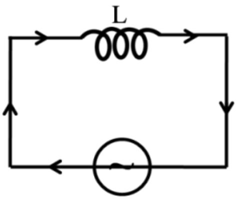
(ii) In a resistance, applied voltage is in phase with the resulting current.

(iii) Apart from instantaneous value, current in the circuit is independent of frequency and decreases with increase in R (similar to that in dc circuits).





(B) An Inductor in an ac Circuit



$E = E_0 \sin \omega t$

$$E - L\frac{dI}{dt} = 0$$

or, $$L\frac{dI}{dt} = E_0 \sin \omega t$$

$$[\text{as } \; E = E_0 \sin \omega t]$$

or, $$\frac{dI}{dt} = \frac{E_0}{L} \sin \omega t$$

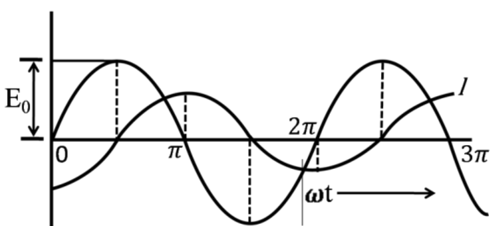
which on integration gives,

$$I = \frac{E_0}{L\omega} \cos \omega t$$

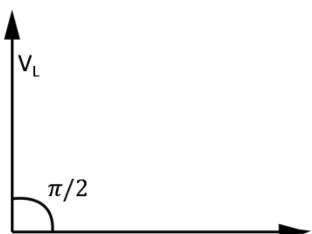
i.e., $$I = I_0 \sin\left(\omega t - \frac{\pi}{2}\right) \quad \text{with } \; I_0 = \frac{E_0}{\omega L}$$

From this expression it is clear that:

(i) The frequency of current in the circuit is same as that of applied emf but current in an inductor lages the applied voltage by (π/2) [or voltage leads the current by (π/2)] as shown in figure.



Wave diagram



Phasor-diagram I_L

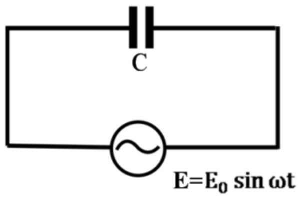
(ii) As here $I_0 = (E_0/\omega L)$, The quantity ωL has the dimensions or resistance as,

$$[\omega L] = \left[\frac{\text{rad}}{\text{sec}} \times \text{H}\right] = \left[\frac{\text{rad}}{\text{sec}} \times \text{ohm} \times \text{sec}\right] = [\text{ohm}]$$

This quantity is called inductive-reactance and is represented by X_L and represents the opposition of a coil to ac, i.e.,

$$X_L = \omega L = 2\pi f L \qquad (\text{as } \omega = 2\pi f)$$

(C) A Capacitor in an ac Circuit



$$E - \frac{q}{C} = 0$$

or, $$q = CE_0 \sin \omega t$$

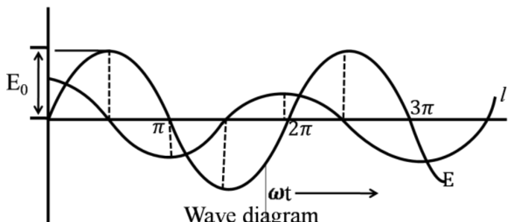
$$(\text{as } E = E_0 \sin \omega t)$$

or, $$I = \frac{dq}{dt} = C\omega E_0 \cos \omega t$$

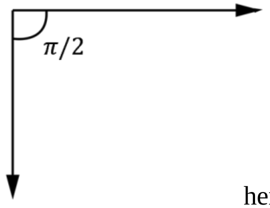
or, $$I = I_0 \sin\left(\omega t + \frac{\pi}{2}\right) \quad \text{with} \quad I_0 = E_0 C\omega$$

From this expression it is clear that:

(i) Current in the circuit has same frequency as the applied voltage but leads it by (π/2) [or voltage across a capacitor lags the current by (π/2)] as shown in figure.



Wave diagram



Phasor-diagram

(ii) As here $I - 0 = C\omega E_0 = \frac{E_0}{X_c}$ with $X_c = \frac{1}{\omega C}$, $(1/\omega C)$ has dimensions of resistance as,

$$\left[\frac{1}{\omega C}\right] = \left[\frac{1}{\text{rads}^{-1}} \times \frac{\text{V}}{\text{coul}}\right] = \left[\frac{\text{V}}{\text{A}}\right] = \text{ohm}$$

And so it represents the opposition of a capacitor to the flow of ac through it and is called capacitive reactance.

(iii) As $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$, with increase in frequency XC decreases non-linearly, i.e., the opposition of a capacitor to ac decreases with increase in frequency. So if $\omega \to 0$, $X_C \to \infty$ and if $\omega \to \infty$, $X_C \to 0$. This is why a capacitor is called high pass-filter and as for dc $\omega \to 0$, $X_C \to \infty$, the opposition of a capacitor to dc is infinite, i.e., a capacitor acts as open circuit in dc circuits in steady-state.

Practice Question online