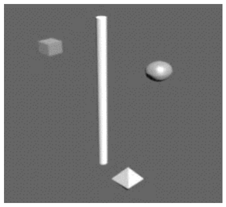
Chapter 12 Rotation

Day - 1

MOMENT OF INERTIA OF A SINGLE PARTICLE

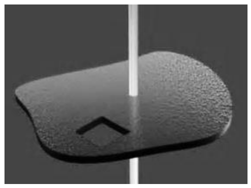
$I = mr^2$

$I = \sum_i m_i r_i^2$



MOMENT OF INERTIA OF A RIGID BODY

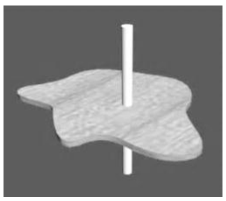
$I = \int r^2 dm$

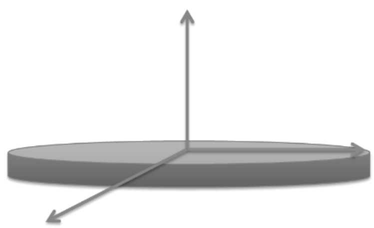


Theorem on Moment of Inertia

Theorem of Parallel Axis

$I = I_{cm} + Mr^2$





Theorem of Perpendicular Axis

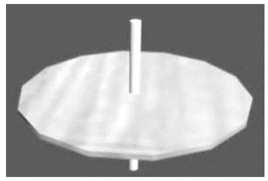
Moment of Inertia is independent of Mass depend on distribution of mass

$I_{disc} = \frac{1}{2}(nM)R^2$

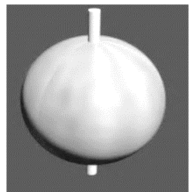
$I_{section} = \frac{1}{2}MR^2$

Moment of inertia of some regular shaped rigid bodies

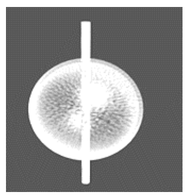
(1)Disc $I = \frac{1}{2}mr^2$



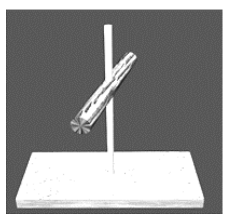
(2)Solid Sphere $I = \frac{2}{5}mr^2$



(3)Hollow Sphere $I = \frac{2}{3}mr^2$

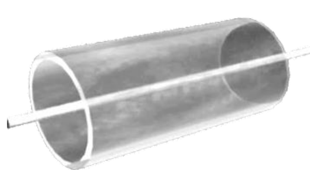


(4) Rod $I = \frac{ml^2}{12}$

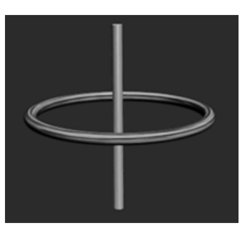


(5)Solid Cylinder $I = \frac{1}{2}mr^2$

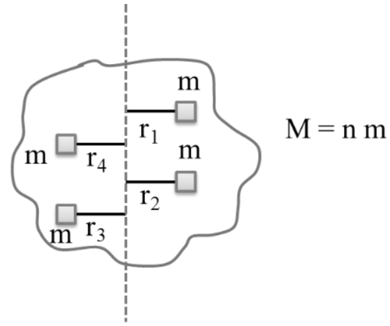
(6)Hollow Cylinderor Ring $I = mr^2$



(7)Ring $I = mr^2$



RADIUS OF GYRATION



Let mass of the rigid body = M

Now $I = \Sigma mr^2$

$I = mr_1^2 + mr_2^2 + - - - - - - + mr_n^2$

$I = mn \frac{(r_1^2 + r_2^2 + - - - - + r_n^2)}{n}$

$MK^2 = M \frac{(r_1^2 + r_2^2 + - - - - + r_n^2)}{n}$

$K = \sqrt{\frac{r_1^2 + r_2^2 + - - - - + r_n^2}{n}}$

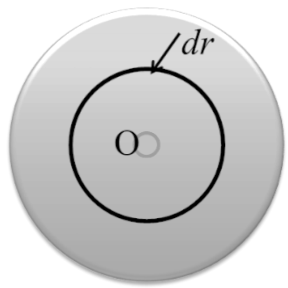
Here K is radius of gyration

RMS value of $r_1, r_2, r_3 \ldots r_n$

Case I

A thin disc of mass M and radius R has mass per unit area $\sigma(r) = kr^2$ where r is the distance from its center. Its moment of inertia about an axis going through its center of mass and perpendicular to its plane is

Given, surface mass density, $\sigma = kr^2$



So, mass of the disc can be calculated by considering small element of area 2πrdr on it and then integration it for complete disc, i.e.

$dm = \sigma\, dA = \sigma \times 2\pi rdr$

$\int dm = M = \int_0^R (kr^2) 2\pi rdr$

$\int dm = M = \int_0^R (kr^2)\ 2\pi rdr$

$\Rightarrow M = 2\pi k \frac{R^4}{4} = \frac{1}{2}\pi kR^4$ ....(i)

Moment of inertia about the axis of the disc,

$I = \int dI = \int dm\, r^2 = \int \sigma dA\, r^2$

$= \int_0^R kr^2\ (2\pi rdr) r^2$

$\Rightarrow\ I = 2\pi k \int_0^R r^5\ \ dr = \frac{2\pi kR^6}{6} = \frac{\pi kR^6}{3}$

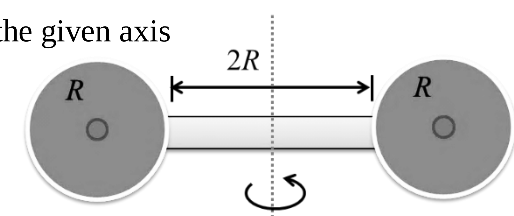
Case II

Two identical spherical balls of mass M and radius R each are suck on two ends of a rod of length 2R and mass M (see figure). The moment of inertia of the system about the axis passing perpendicularly through the center of the rod is Key idea This problem will be solved by applying parallel axis theorem, which states that moment of inertia of a rigid body about any axis is equals to its moment of inertia about a parallel axis through its center of mass plus the product of the mass of the body the square of the perpendicular distance between the axis.

We know that moment of inertia (MI) about the principle axis of the sphere is given by

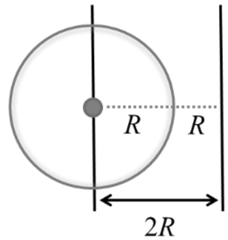
$I_{sphere} = \frac{2}{5}\ MR^2$

Using parallel axis theorem, moment of inertia about the given axis



$I_1 = \frac{2}{5}\ MR^2 + M(2R)^2$

$I_1 = \frac{22}{5} MR^2$



Considering both sphere at equal distance form the axis, moment of inertia due to both sphere about this axis will be

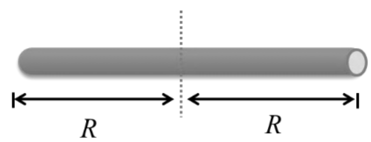
$2I_1 = 2 \times \frac{22}{5} MR^2$

Now, moment of inertia of rod about its perpendicular bisector axis is given by

$I_2 = \frac{1}{12} ML^2$

Here, given that

L = 2R



$\therefore \; I_2 = \frac{1}{12} M\,(2R)^2 = \frac{1}{3} MR^2$

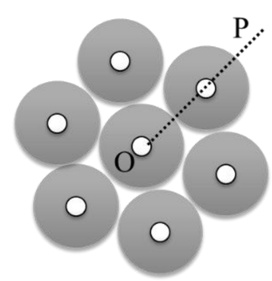
So, total moment of inertia of the system is

$I = 2I_1 + I_2 = 2 \times \frac{22}{5} MR^2 + \frac{1}{3} MR^2$

$\Rightarrow I = \left(\frac{44}{5} + \frac{1}{3}\right) MR^2 = \frac{137}{15} MR^2$

Case III

Seven identical circular planner discs, each of mass M and radius R are welded symmetrically The moment of inertia of the arrangement about an axis normal to the plane and passing through the point P is



From theorem of parallel axis,

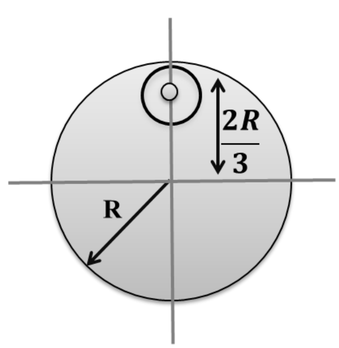
$I = I_{cm} + 7m\,(3R)^2$

$= \left[\frac{MR^2}{2} + 6 \times \left\{\frac{MR^2}{2} + M(2R)^2\right\}\right] + 7M(3R)^2 = \frac{181MR^2}{2}$

Case IV Cavity

Form a uniform circular disc of radius R and 9M, a small disc of radius $\frac{R}{3}$ is removed The moment of inertia of the remaining disc about an axis perpendicular to the plane of the disc and passing through center of disc is

$I_{Remaining} = I_{total} - I_{Cavity}$



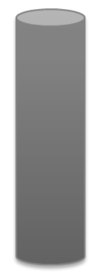
Case V

The moment of inertia of a uniform cylinder of length l and radius R about its perpendicular bisector is I. what is the ratio l/R such that the moment of inertia is minimum?

MI of a solid cylinder about its perpendicular bisector of length is

$$I = M\left(\frac{l^2}{12} + \frac{R^2}{4}\right)$$

$$\Rightarrow I = \frac{mR^2}{4} + \frac{ml^2}{12} = \frac{m^2}{4\pi\rho l} + \frac{ml^2}{12} \qquad [\because\ \rho\pi r^2 l = m]$$



For I to be minimum,

$$\frac{dI}{dl} = -\frac{m^2}{4\pi\rho}\left(\frac{1}{l^2}\right) + \frac{ml}{6} = 0$$

$$\Rightarrow \frac{m^2}{4\mu\pi\rho} = \frac{ml^3}{6} \Rightarrow l^3 = \frac{3m}{2\pi\rho}$$

$$\Rightarrow l = \left(\frac{3}{2}\right)^{\frac{1}{3}}\left(\frac{m}{\pi\rho}\right)^{1/3}$$

$$\rho = \frac{m}{\pi R^2 l} \quad \Rightarrow R^2 = \frac{m}{\pi\rho l}$$

$$\Rightarrow R^2 = \frac{m}{\pi\rho}\left(\frac{2}{3}\right)^{\frac{1}{3}}\left(\frac{\pi\rho}{m}\right)^{\frac{1}{3}} = \left(\frac{m}{\pi\rho}\right)^{\frac{2}{3}} \ \left(\frac{2}{3}\right)^{1/3}$$

$$\Rightarrow \quad R = \left(\frac{m}{\pi\rho}\right)^{\frac{1}{3}}\left(\frac{2}{3}\right)^{\frac{1}{6}}$$

$$\frac{l}{R} = \frac{\left(\frac{3}{2}\right)^{1/3}\left(\frac{m}{\pi\rho}\right)^{1/3}}{\left(\frac{m}{\pi\rho}\right)^{\frac{1}{3}}\left(\frac{2}{3}\right)^{1/6}} = \left(\frac{3}{2}\right)^{1/3} + \left(\frac{3}{2}\right)^{1/6}$$

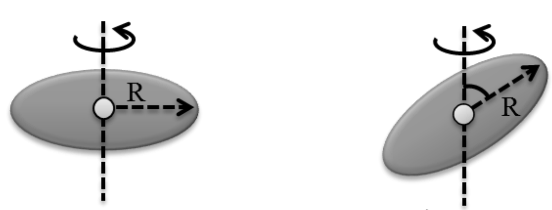
$$\therefore \quad \frac{l}{R} = \sqrt{\frac{3}{2}}$$

Case:- Moment of Inertia

Rod → mass = m, Length = L

$I = \frac{ml^2}{3}$ $\qquad I = \frac{mL^2}{12}$ $\qquad I = \frac{mL^2 \sin^2\theta}{3}$ $\qquad I = \frac{mL^2}{12}\sin^2\theta$

Disc → mass = m, Radius = R



$I = \frac{1}{2}mR^2$ $\qquad I = \frac{1}{2}mR^2 \sin^2\theta$

Multiple rods

Each rod of mass m and length L

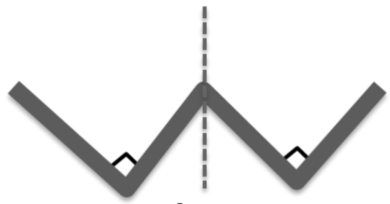
$I = I_1 + I_2 + I_3 + I_4$

$= 2(I_1 + I_2)$

$I_1 = I_4 = + \left[\sqrt{l + \frac{l}{2}}\right]^2$

$I_2 = I_3, \quad I_2 = I_4$

$I_2 = I_3 = \frac{m}{3} L^2$



$I = \frac{10\ ml^2}{3}$