Chapter 9 Graphical Transformation

Day 1

Some Standard Graphs

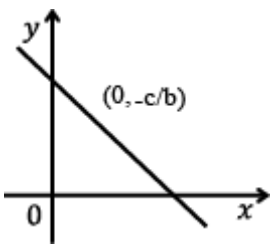
Straight Line

(a)

$ax + by + c = 0$

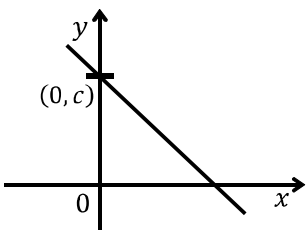
when $x = 0$, $y = -c/b$

when $y = 0$, $x = -c/a$

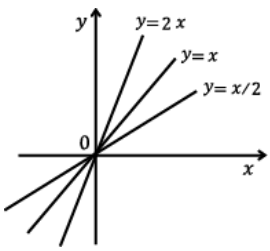


(b)

$y = mx + c$



$y = mx$



Quadratic Parabola

The graph of $y = ax^2 + bx + c$ is symmetric about the vertical line $x = -b/2a$ and it's vertex is given by $\left(\frac{-b}{2a}, -\frac{b^2-4ac}{4a}\right)$ $i.e.\left(\frac{-b}{2a}, \frac{D}{4a}\right)$

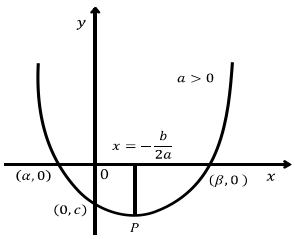
If $a > 0$, the parabola opens upward while.

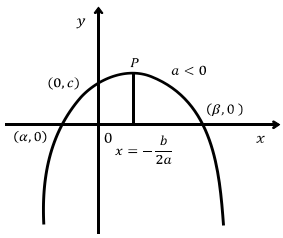
If $a < 0$, the parabola opens downward.

Find x when $y = 0$,

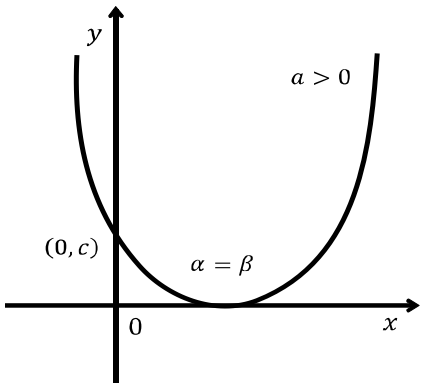
$$(\alpha, \beta) = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

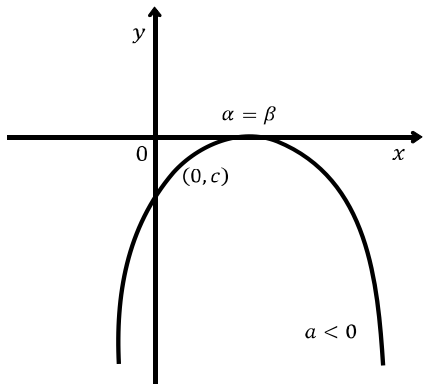
(i) When roots are real and unequal (D > 0)



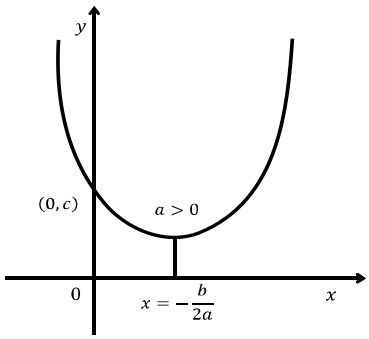


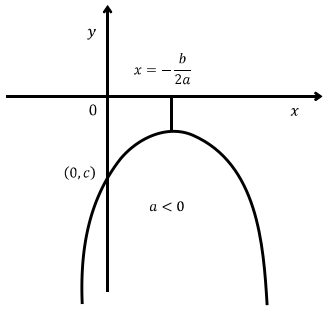
(ii) When roots are equal (D = 0)





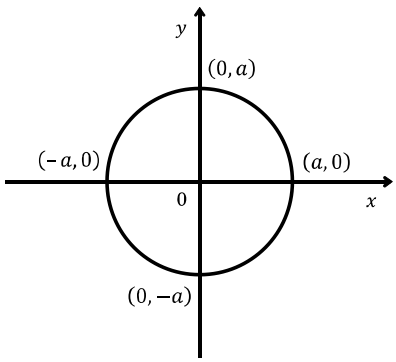
(iii) When roots are imaginary (D < 0)





Circle

(i) $x^2 + y^2 = a^2$ is a circle centred at (0,0)
Having radius = a
$when\ y = 0, x = \pm a when\ x = 0, y = \pm a$



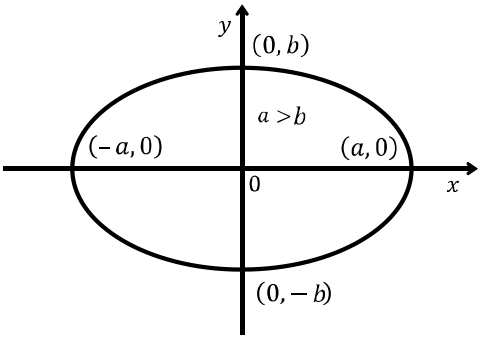
(ii) $x^2 + y^2 + 2gx + 2fy + c = 0$ is a circle centred at *(-g, -f)*
$having\ radius = \sqrt{g^2 + f^2 - c}$

Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$when\ x=0, y=\pm b$

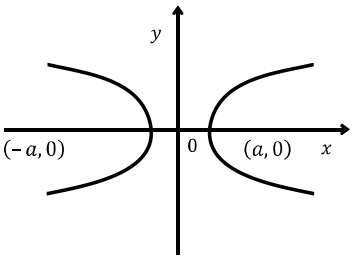
$when\ y=0, x=\pm b$

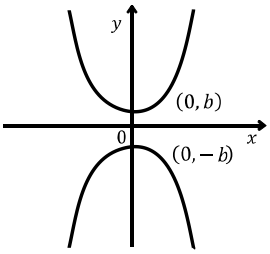


Hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

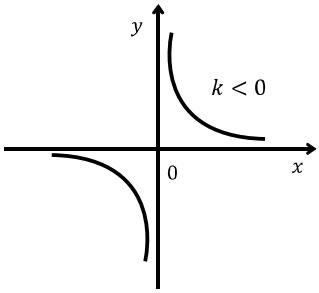
$when\ x=0, y^2=-b^2$

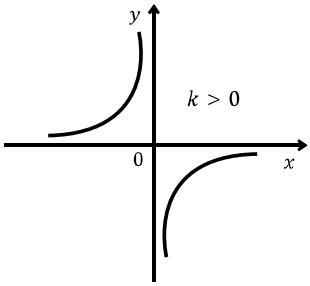




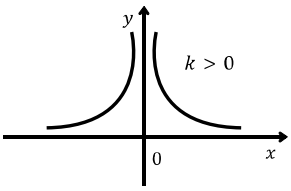
Rectangular Hyperbola

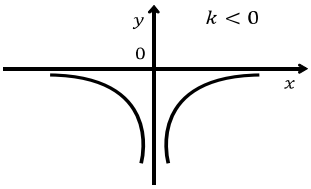
$xy=k$





$x^2y=k$



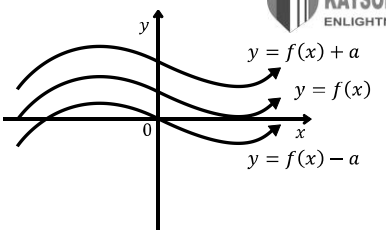


Transformation of Graphs

(a)

$y = f(x)$ to transform $f(x) \pm c$

where c > 0

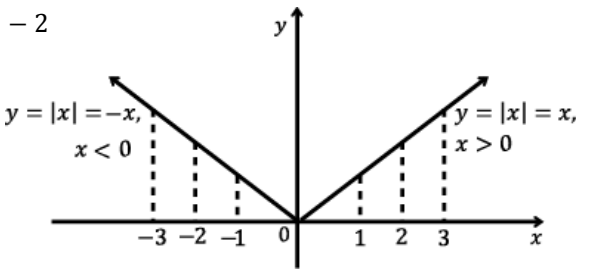


Illustration

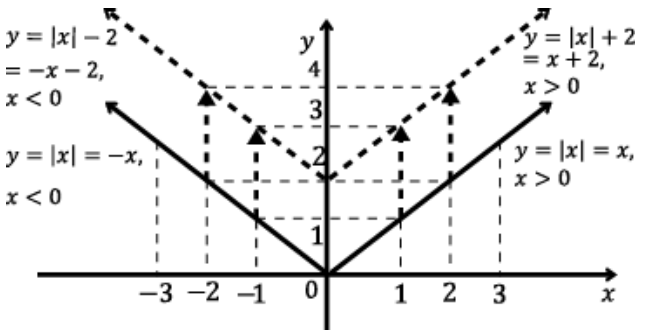
Plot $y = |x|, y = |x| + 2$ and $y = |x| - 2$

Solution

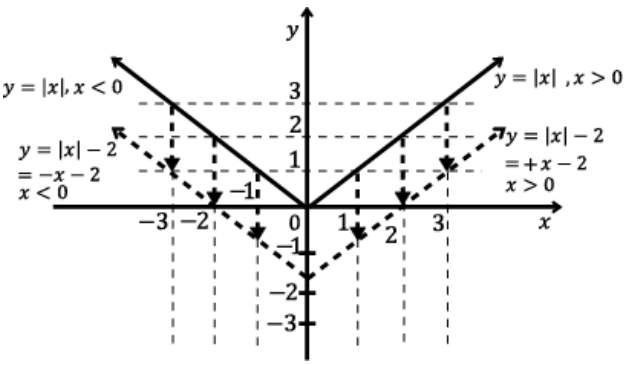
We know
$y = |x|$
(modulus function)
could be
plotted as



$y = |x| + 2$ is shifted
upwards by
2 units



Also
$y = |x| - 2$ is shifted
downwardsby 2 units.



(b)

(i) $f(x)$ transforms to $f(x-a)$

$$f(x) \rightarrow f(x-a), \text{ a is positive}$$

Shift the graph of $f(x)$ through 'a' towards right.

(ii) $f(x) \rightarrow f(x+a)$, a is positive.

Shift the graph of $f(x)$ through 'a' towards left.

Illustration

$$Ploty = |x| and\ y = |x-2|$$

Solution

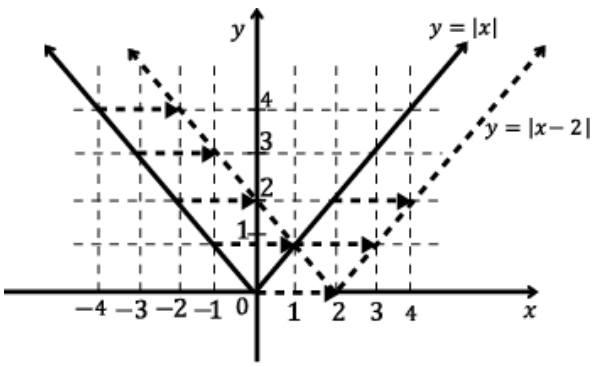
As discussed

$f(x) \rightarrow f(x-a)$,

Shift the graph of $f(x)$ through 'a' towards right.

$\Rightarrow y = |x-2|$

is shifted '2' units towards right.



(c)

$(i) f(x) transforms\ t\ \{af(x)\} where\ a > 1$

$f(x) \rightarrow af(x), a > 1$

Stretch the graph of $f(x)$ **'a' times along y-axis.**

$(ii) f(x) \rightarrow \frac{1}{a} f(x), a > 1$

Shrink the graph of $f(x)$ **'a' times along y-axis.**

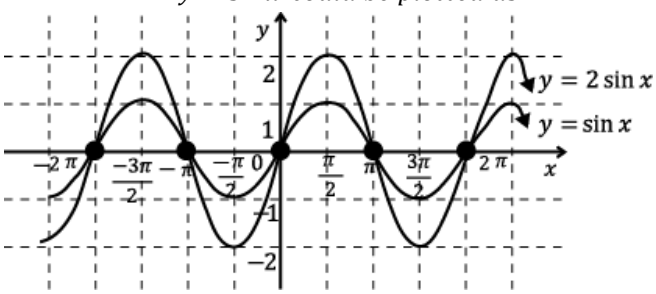
Illustration

$$Plot\ y = \sin x\ and\ y = 2\sin x$$

Solution

We know

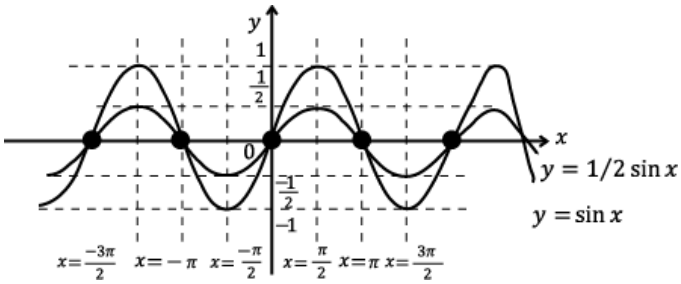
$$y = \sin x\ could\ be\ plotted\ as$$



Illustration

$$Plot\ y = \sin x\ and\ y = 1/2 \sin x.$$

Solution



(d)

$(i)\boldsymbol{f(x)transforms\ to\ f(ax), a} > 1.$

$i.e. f(x) \to f(ax), a > 1.$

$shrink\ the\ graph\ of\ f(x)'\boldsymbol{a'times\ along\ x - axis}.$

$(ii) f(x) transforms\ to\ f\left(\frac{x}{a}\right), a > 1$

$i.e. f(x) \to f\left(\frac{x}{a}\right), a > 1$

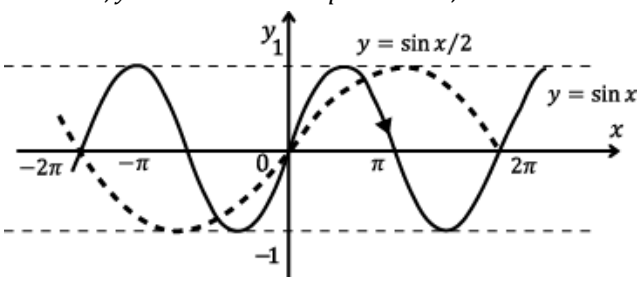
$stretch\ the\ graph\ of\ f(x)'a'times\ along\ x - axis.$

Illustration

$$Plot\ y = \sin x\ \ and\ y = \sin x/2$$

Solution

$we\ know, y = \sin x\ could\ be\ plotted\ as;$

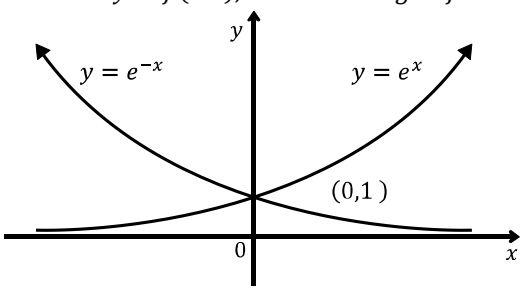


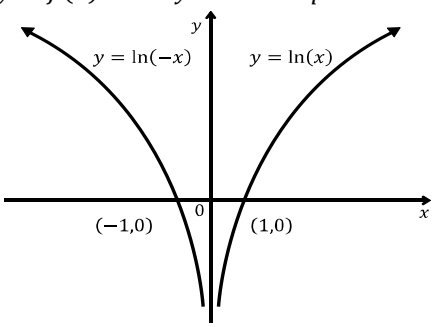
(e)

$\boldsymbol{To\ draw\ graph\ of\ y = f(-x) when\ f(x) is\ given}:$

$i.e. f(x) \to f(-x)$

$To\ draw\ y = f(-x), take\ the\ image\ of\ the\ curve\ y = f(x) in\ the\ y - axis\ as\ plane\ miror..$



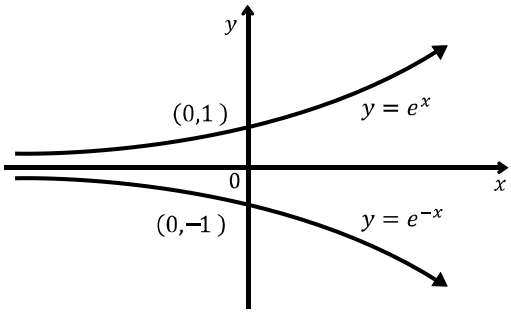


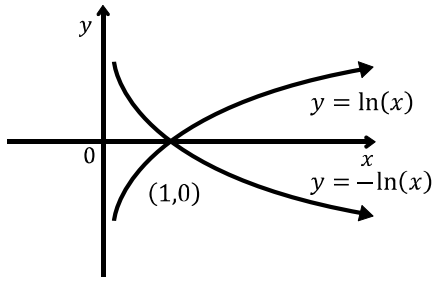
(f)

$\boldsymbol{To\ draw\ y = -f(x) when\ y = f(x)\ is\ given}$

$i.e. f(x) \to -f(-x)$

$To\ draw\ y = -f(x)\ take\ image\ of\ f(x) in\ the\ x - axis\ as\ plane\ miror.$





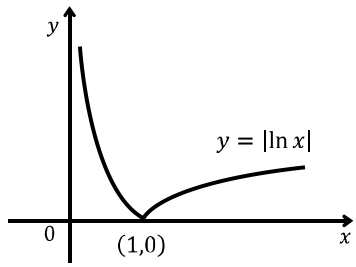
(g)

$\boldsymbol{y = |f(x)|}$

$|f(x)| = f(x),$

$if\ f(x) \geq 0$

$-f(x)\ \ if\ f(x) < 0$



(h)

$\boldsymbol{To\ draw\ y = f(|x|)\ when\ f(x) is\ given}$

$y = f(x) = \begin{cases} f(x), if\ x \geq 0 \\ f(-x), if\ x \leq 0 \end{cases}$

$On\ the\ right\ of\ y - axis.$

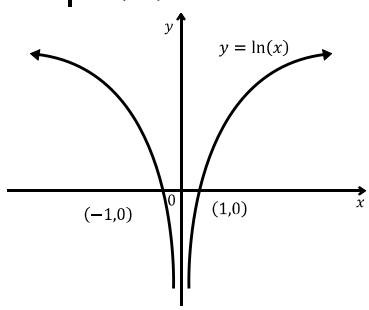
$Plot\ the\ graph\ of\ f(x) as\ such$

$on\ the\ left\ of\ y - axis$

$Plot\ the\ mirror\ image\ of\ the\ graph\ of\ f(x)$

$(of\ the\ portion\ lying\ on\ right\ of\ y - axis)$

$Note. f(|x|)\ is\ always\ as\ even\ function.$



(i)

$\boldsymbol{To\ draw\ y = [f(x)\ ], when\ the\ graph\ of\ y = f(x) is\ given}$

To draw $= [f(x)]$, where [.] denotes greatest integral function. {i.e., now y can not be fraction}.

Here in order to draw $y = [f(x)]$ mark the integer on *y-axis* . Draw the horizontal lines through integers till they intersect the graph. Draw vertical dotted lines from these intersection points.

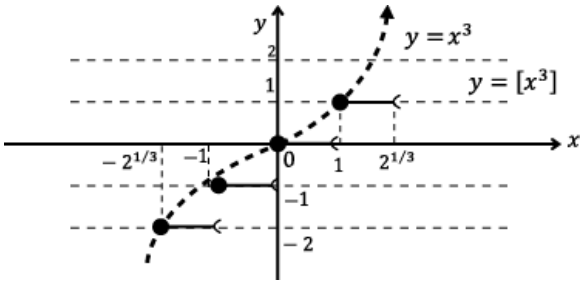
Finally draw horizontal lines parallel to *x- axis* from any intersection point to the nearest vertical dotted line with blank dot at right end in case $f(x)$ increases.

Illustration

$$Draw\ the\ graph\ of\ y = [x^3]$$

Solution

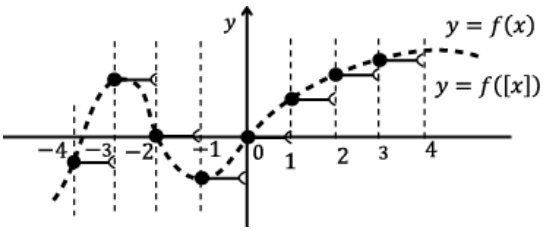
(a) Draw horizontal lines through integers till they intersect the graph.
(b) Draw vertical dotted lines from these intersection points.
Finally draw horizontal lines parallel to x-axis from any intersection point to the nearest vertical dotted line with blank dot at right end.



(j)

***To draw $y = f([x])$ when the graph of $y = f(x)$ is given*:**

Mark the integers on the x-axis. Draw vertical lines till they intersect the graph of $f(x)$. From these intersection points draw horizontal lines (parallel to x-axis) to meet the nearest right vertical line with a blank dot in each nearest right vertical line which can be shown as in the figure.



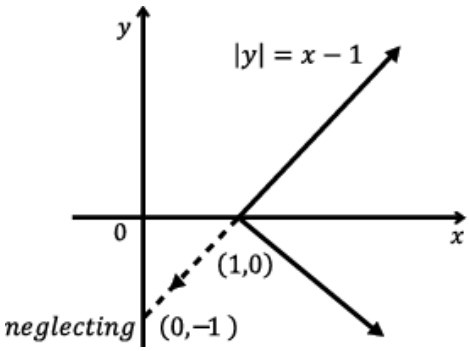
(k)

Draw the graph of $|y| = f(x)$

when $y = f(x)$ is given

(i) Remove the portion of the graph which lies below x-axis. (the equation $|y| = f(x)$ is not satisfied when *f(x)* is negative).

(ii) Plot the remaining portion of the graph and also its mirror image in the x-axis (when $f(x) > 0$ then $y = \pm f(x)$.



(*l*)

(i)When $f(x)$and $g(x)$ are given then find,

$h(x) = max.\{f(x), g(x)\}$

$$h(x) = \begin{cases} f(x) & ,when\ f(x) > g(x) \\ g(x) & ,when\ f(x) < g(x) \end{cases}$$

∴ sketch f(x) when its graph is above the graph of g(x)and sketch g (x) and sketch g(x) when its graph is

Draw graph of $y = \max\{2x, x^2\}$ above the graph of f(x).

Illustration

Draw graph for $y = \max.\{2x, x^2\}$ and discuss the continuity and differentiability.

Solution

Here, to draw

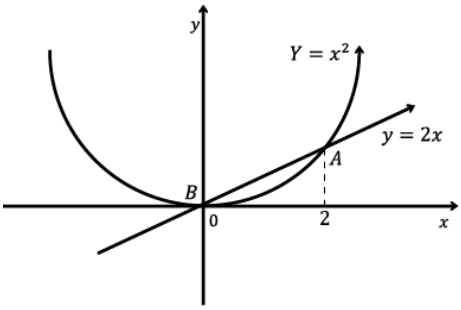
$$y = max.\{2x, x^2\}$$

Firstly plot $y = 2x$ and $y = x^2$ on graph and put

$$2x = x^2 \Rightarrow x = 0, 2$$

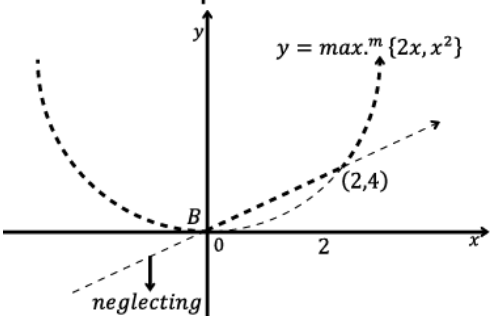
$(i.e., their\ point\ of\ intersection)$.

Now since $y = max.(2x, x^2)$ we have to neglect the curve below point of intersections thus, the required graph is, as shown.



Thus, from the given graph $y = max.(2x, x^2)$ we can say $y = max.(2x, x^2)$ is continuous for all $x \in R$.

But $y = max.\{2x, x^2\}$ is differentiable for all $x \in R - \{0,2\}$



Illustration

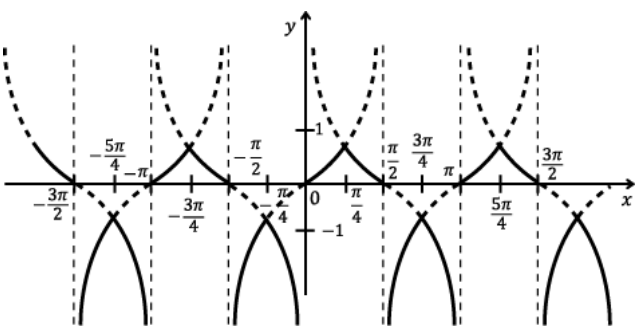
$$Draw\ f(x) = \min\{tanx, \cot x\}$$

Solution

First plot both $f(x) = \tan x\ and\ g(x) = \cot x$ by a dotted curve as can be seen from the graph and then darken those dotted lines for which $f(x) < g(x)\ and\ g(x) < f(x)$.

As from the graph we have,

$$min.\{\tan x, \cot x\} = \begin{cases} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \tan x, \ -\frac{\pi}{2} < x \le -\frac{\pi}{4} \\ \cot x, \ -\frac{\pi}{4} \le x < 0 \\ \tan x, \ 0 \le x \le \frac{\pi}{4} \\ \cot x, \ \frac{\pi}{4} \le x \le \frac{\pi}{2} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$



How to draw graph of Polynomial

1. Factories the function, if possible
2. Find the roots i.e. put y = 0 and get the values of x.
3. Find point of local maxima/minima.
4.Finally find out $\lim_{x\to\infty} y$ & $\lim_{x\to-\infty} y$ & it would five rough sketch of polynomial.

Illustration

$$Graph\ y = (x-1)(x-3)^2$$

Solution

Step 1

$$y = 0$$
$$\Rightarrow x = 1, x = 3$$

are roots of function where graphs cuts x-axis

$$f(0) = (-1)(-3)^2 = -9$$

hence graph cuts y-axis at pt $(0, -9)$

Step 2

$$\frac{dy}{dx} = 2(x-1)(x-3) + (x-3)^2 = 0$$
$$\Rightarrow (x-3)(2x-2+x-3) = 0$$
$$\Rightarrow (x-3)(3x-5) = 0$$

Critical pts are $x = 3,, 5/3$

$$\frac{d^2y}{dx^2} = (x-3).3 + (3x-5)$$
$$\Rightarrow 6x - 14$$
$$\left(\frac{d^2y}{dx^2}\right)_{x=3} = +ve$$

hence local minima at $x = 3$

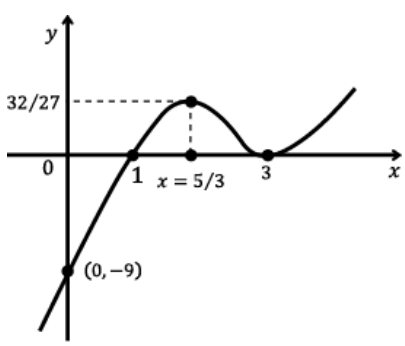
$$y_{min} = 0$$
$$\left(\frac{d^2y}{dx^2}\right)_{x=5/3} = -ve$$

hence local maxima at $x = 5/3$

$$(y_{max}) = \frac{2}{3} \times \frac{16}{9} = \left(\frac{32}{27}\right)$$

Step 3

$$\lim_{x\to-\infty} (x-1)\,(x-3)^2 = -\infty$$



From the above information we draw the graph as follows.

Illustration

$$Sketch\ the\ function\ y = \frac{2}{1+x^2}$$

Solution

Step 1

Domain of the function is $(-\infty, \infty)$

$\delta(0) = 2$ and function is symmetrical about y-axis

Step 2

$$\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2} = 0$$

Critical pt is $x = 0$

$$\frac{dy}{dx} > 0 \; wher \; x < 0 \; (y \; increasing)$$
$$\frac{dy}{dx} < 0 \; when \; x > 0 (y \; decreasing)$$

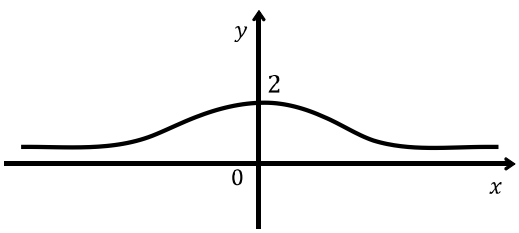
Point $x = 0$is point of maxima.

$(y)_{x=0} = 2$

Step 3

$$\lim_{x \to -\infty} y = 0$$
$$\lim_{x \to -\infty} y = -0$$

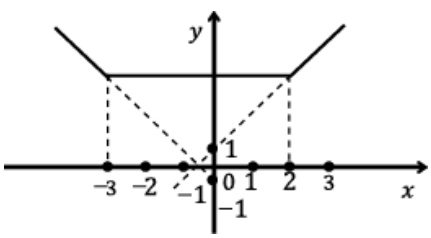
So graph of the function is



<u>Draw the graph</u>

$(\boldsymbol{i})\boldsymbol{y} = |\boldsymbol{x}+\mathbf{3}| + |\boldsymbol{x}-\mathbf{2}|$

$$f(x) = |x+3| + |x-2|$$
$$\left.\begin{array}{l} \Rightarrow -(x+3)-(x-2) \\ \Rightarrow -(2x+1) \end{array}\right\} when \; x < -3$$
$$\left.\begin{array}{l} f(x) = (x+3)-(x-2) \\ \quad \Rightarrow 5 \end{array}\right\} -3 \le x < 2$$
$$\left.\begin{array}{l} f(x) = (x+3)+(x-2) \\ \quad \Rightarrow 2x+1 \end{array}\right\} when \; x \ge 2$$



So,

$$f(x) \begin{cases} = -2x-1 & x < -3 \\ = \quad 5 & -3 \le x < 2 \\ = 2x+1 & x \ge 2 \end{cases}$$

$(\boldsymbol{ii})|\boldsymbol{y}| = (\boldsymbol{x}-\mathbf{1})(\boldsymbol{x}-\mathbf{3})$

$\because |y| = (x-1)(x-3)$

The graph must be same for +ve as well as –ve values of y.i.e.

If

$y = 2$

$\Rightarrow (x-1)(x-3)$

and

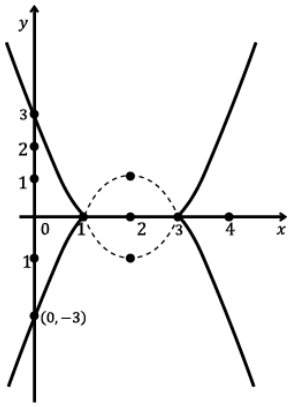
$\Rightarrow 2$

$y = -2$

Then

$\Rightarrow (x-1)(x-3)$

$\Rightarrow 2$



Which clearly shows that this graph will be symmetrical about x-axis so we first draw graph of function $y = (x-1)(x-3)$ and take image of this curve on x-axis (treating as minor)

But when $|< x < 3|$

Then the expression (x-1)(x-3) will have –ve sign but $|y|$ can never be negative so there will be no portion of graph lying in interval (1,3)

$(\boldsymbol{iii})|\boldsymbol{y}| = |(\boldsymbol{x}-\mathbf{1})(\boldsymbol{x}-\mathbf{3})|$

Since we have already drawn the graph of

$y = |(x-1)(x-3)|$

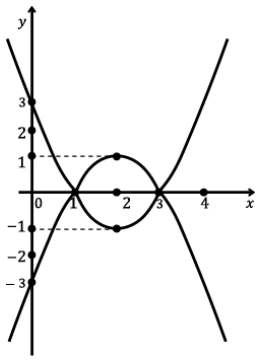
So, graph of

$|y| = |(x-1)(x-3)|$

Will also contain the image of curve

$y = |(x-1)(x-3)|$

On the x-axis treating it as a mirror.

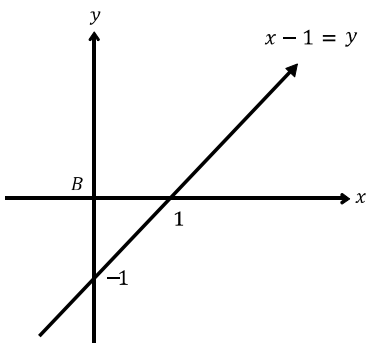


Illustration

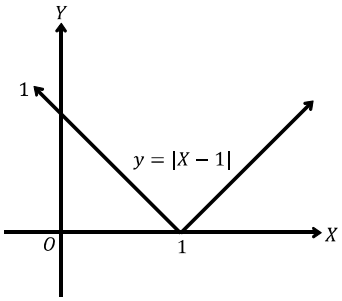
$Draw\ the\ graph\ for\ y = \big|2 - |x-1|\big|.$

Solution

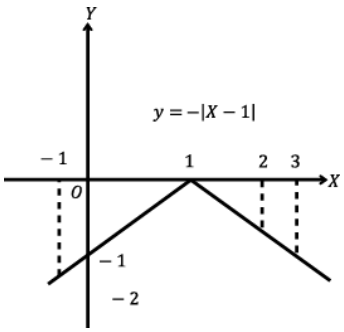
Here $y = (x-1)$can be plotted



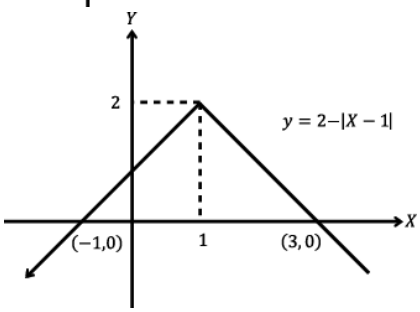
$\Rightarrow y = |x - 1|$can be plotted



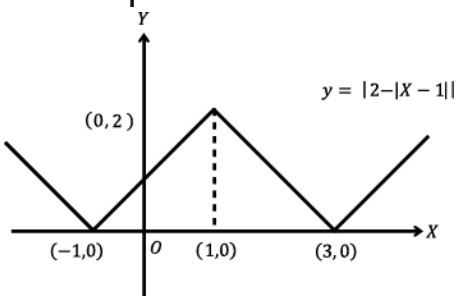
$\Rightarrow y = -|x - 1|$ can be plotted



$\Rightarrow y = 2 - |x - 1|$ can be plotted



Thus $y = \left|2 - |x - 1|\right|$ can plotted



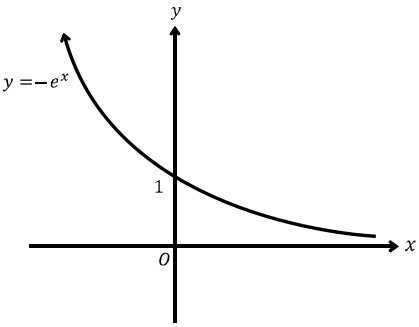
Note: From above figure we could say $y = \left|2 - |x - 1|\right|$ is not differentiable at $x = \{-1,1,3\}$ as sharp edges t $x = -1,1,3$.

Illustration

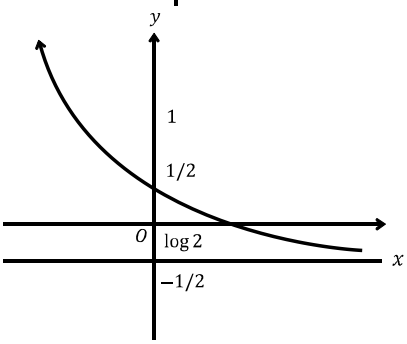
$$Sketch\ the\ graph\ for\ y = \left|e^{-|x|} - \frac{1}{2}\right|$$

Solution

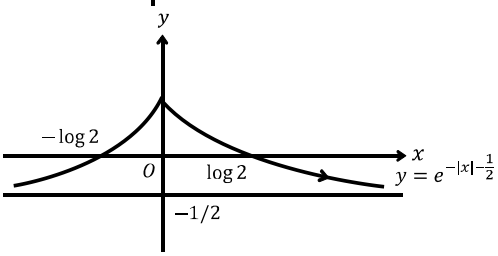
As we know the graph for $y = e^{-x}$



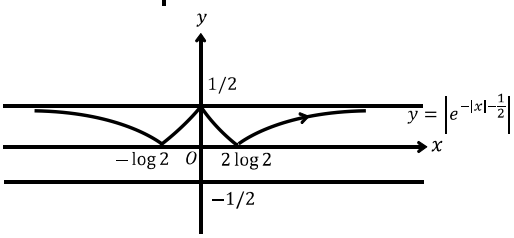
$(i)\ y = e^{-x} \rightarrow y = e^{-x} - \frac{1}{2}$



$(ii) y = e^{-x} - \frac{1}{2} \rightarrow y = e^{-|x|} - \frac{1}{2}$



$(iii)\ y = \left|e^{-|x|} - \frac{1}{2}\right|$



When $f(x)$ and $g(x)$ are two functions and are transformed to their sum.

$$f(x), g(x) \rightarrow f(x) + g(x) = h(x)$$

There is no direct approach, but we can use following steps.

Step1 : Check when $g(x) = 0 \Rightarrow h(x) = f(x)$

Step 2: When $g(x) > 0$ then $h(x) > f(x)$ i.e. the graph of $h(x)$ lies above the graph of $f(x)$.

Step 3: When $g(x) < 0$ then $h(x) < f(x)$ the graph of $h(x)$ lies below the graph $f(x)$.

Illustration

$$Ploty = x + \sin x$$

Solution

Here

$y = f(x) + g(x)$

Where

$f(x) = x \; and \; g(x) = \sin x$

Here

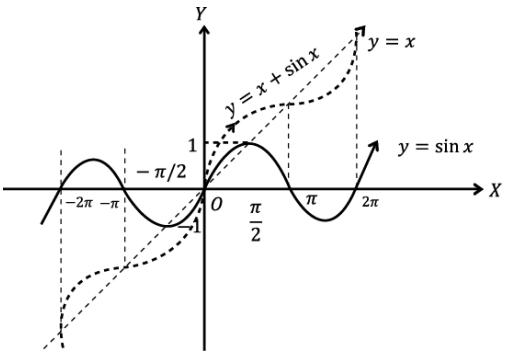
$g(x) = 0 \Rightarrow y = x$

$also \; when$

$g(x) > 0 \Rightarrow x + \sin x > x$

$and \; when$

$g(x) < 0 \Rightarrow x + \sin x < x$



<u>Draw the graph for $y = f(x) \sin x$ *where graph of* $y = f(x)$ *is given*</u>

Clearly

$$-f(x) \le f(x).sinx \; \le f(x)$$

$$\{as - 1 \le \sin x \le 1\}$$

Hence, graph for $y = f(x).\sin x$ would be lying between the graph of $y = f(x)$ and $y = -f(x)$. It amounts to just drawing graph of sin x in between the graphs of $y = \pm f(x)$.

Illustration

$$Draw \; grpah \; of \; y = xsin \; x$$

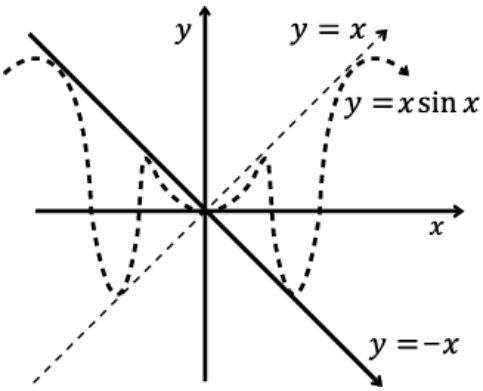
Solution

Here

$y = x \sin x$

$\Rightarrow -x \le y \le x[as - 1 \le \sin x \le 1]$

Thus, it can be plotted as shown.



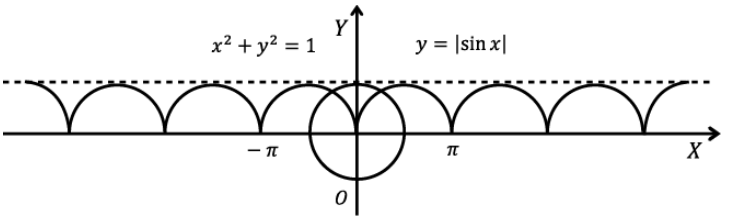
Illustration

Find the number of solutions of the equations

$$y = |\sin x| and \; x^2 + y^2 = 1.$$

Solution

To Find the number of solutions of two curves we should find the point of intersection of two curves.
As the know



$x^2 + y^2 = 1$ is circle and $y = |\sin x|$ is the image of –ve values of $y = \sin x$ about x axis. Thus we can plot them as.
Which shows the two curves intersects at two points.
Number of solutions is 2.

<u>Plotting graph of $f(x - [x])$</u>

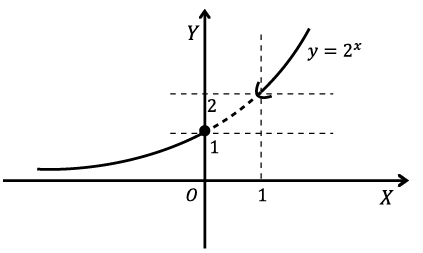
Graph of $f(x - [x])$ can be obtained from and the graph of $f(x)$ by following rule.
"Retain the graph of $f(x)$ for values of x lying between interval $[0,1)$. Now it can be repeated for rest of points. New obtained function is periodic with period 1."

Illustration

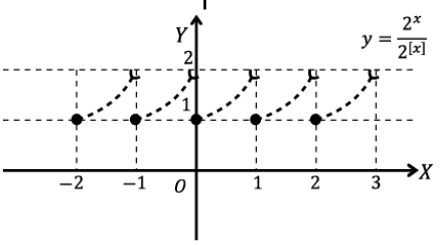
$$Sketch\ the\ graph\ of\ y = \frac{2^x}{2^{[x]}}$$

Solution

As we know 2^x is exponential function and we want to transform it to $2^{x-[x]}$, it retain the graph for $x \in [0,1)$ and repeat for rest points.



Here to retain graph between $x \in [0,1)$, so we get.



Question Practice Online

Test Paper

$Ques. 1. Given\ the\ system\ of\ straight\ lines\ a(2x+y-3)+b(3x+2y-5)=0,$
$that\ line\ of\ the\ system\ situated\ farthest\ away\ the\ point\ (4,-3)\ has\ the$
$equation\ is$

(a) $4x+11y-15=0$ (b) $7x+y-8=0$
(c) $4x+3y-7=0$ (d) $3x-4y+1=0$

$Ques. 2. If\ the\ wquation\ e^{||x|-2|+b}=2\ has\ four\ solution\ then\ b\ lies\ in$

(a) $(\ln 2\ ,-\ln 2)$ (b) $(\ln 2-2\ ,\ln 2)$
(c) $(-2,\ln 2)$ (d) $(0\ ,\ln 2)$

$Ques. 3. Let\ f(x)=\frac{\sin x}{x}, where\ 0<x<\frac{\pi}{2}. Then$

(a) $\sin^2 x < x\sin(\sin x)$ (b) $\sin^2 x > x\sin(\sin x)$
(c) $\sin^2 x > 1+x\sin(\sin x)$ (d) none of these

$Ques. 4. \lim_{x}\left[\frac{\sin s_{gm}(x)}{s_{gm}(x)}\right], where\ [.]\ denotes\ greatest\ integral\ function.$

(a) 0 (b) 1
(c) -1 (d) does not exists.

$Ques. 5. If\ f(x)=\frac{a\cos x-\cos bx}{x^2}, x\neq 0\ and\ f(0)=4\ is\ continuous\ at$
$x=0\ then$
$the\ dordered\ pair\ (a,b)\ is\ ;$

(a) $(\pm 1,3)$ (b) $(1,\pm 3)$
(c) $(-1,-3)$ (d) $(1,3)$

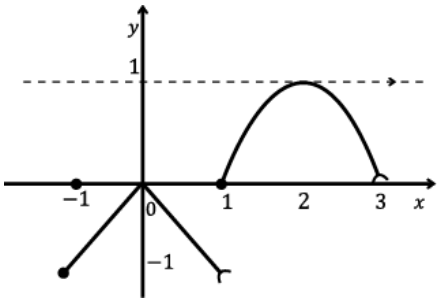
$Ques. 6. Total\ number\ of\ ways\ of\ factorising\ the\ number\ of\ 676\ in\ two$
$factors, that\ are\ prime\ to\ each\ other\ is\ equal\ to$

(a) 100 (b) 50
(c) 25 (d) none of these

$Ques. 7. Let\ f(x)=\left\{\frac{\sin 5x}{2x}\right\}, where\{.\}\ denotes\ the\ fractional\ part\ of\ x. Then$
$\lim_{x\to 0} f(x)\ is$

(a) 1/3 (b) 1/5
(c) 1/2 (d) none of these

$Ques. 8. For\ the\ graph\ of\ the\ function\ y=f(x)\ , the\ valid\ statement\ is$



(a)f(x) is continuous when x$\epsilon[-1,1)[1,3)$
(b)f(x) is continuous when x$\epsilon[-1,1) \cup [1,3)$
(c)f(x) is not differentiable at x = {0,1,3}
(d)f(x) is not differentiable at x = {0,1}

$Ques. 9. If\ the\ line\ x\cos\alpha + y\sin\alpha = P\ is\ a\ normal\ to\ the\ curve$
$(x+a)y = c^2, then$

(a) $0 < \alpha < \pi/2$ (b)$\pi < \alpha < 3\pi/2$
(c) $\pi/2 < \alpha < \pi$ (d) none of these

$Ques. 10. On\ the\ intreval\ [5\pi/4\,, 4\pi/3], the\ least\ value\ of the\ function$

$$f(x) = \int_{5\pi/4}^{x} (3\sin t + 4\cos t)\,dt\ is$$

(a) $\frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}$ (b) $\frac{3}{2} - \frac{1}{\sqrt{2}} + 2\sqrt{3}$
(c) $\frac{3}{2} - \frac{1}{\sqrt{2}} - 2\sqrt{3}$ (d) none of these

$Ques. 11. Fundamental\ period\ of\ the\ function,$

$$f(x) = \cos(\tan x + \cot x)\,.\cos(\tan x - \cot x)\,, is$$

(a) $\pi/4$ (b) $\pi/2$
(c) π (d) 2π

$Ques. 12. The\ number\ of\ critical\ points\ of$
$f(x) = max.\{\sin x, \cos x\} \forall\ x\epsilon(-2\pi, 2\pi)$

(a) 5 (b) 6
(c) 7 (d) 8

$Ques. 13. The\ number\ of\ bijective\ functions\ f: A \to A, where\ A = \{1,2,3\}$
$such\ that\ f(1) \neq 3, f(2) \neq 1, f(3) \neq 2\ is$

(a) 1 (b) 2
(c) 9 (d) none of these

$Ques. 14. The\ angle\ at\ which\ the\ curve\ y = ke^{kx} intersects\ the\ y - axis\ is$

(a) $\tan^{-1}(k^2)$ (b) $\cot^{-1}(k^2)$

(c) $\sin^{-1}\left(1/\sqrt{1+k^4}\right)$ (d) $\sec^{-1}\left(\sqrt{1+k^4}\right)$

Ques. 15. *The parabolas* $y^2 = 4ax$ *and* $x^2 = 4by$ *intersect orthogonally at point* $P(x_1, y_1)$ *where* $x_1, y_1 \neq 0$ *then*

(a) $b = a^2$ (b) $b = a^3$

(c) $b^3 = a^2$ (d) none ofthese

Ques. 16. *If at* $x = 1, y = 2x$ *is tangent to the parabola* $y = ax^2 + bx + c$, *then respective values of* $a, b,$ *are*

(a) $1/2, 1, 1/2$ (b) $1, 1/2, 1/2$

(c) $1/2, 1/2, 1$ (d) none of these

Ques. 17. *Let* x *and* y *be real numbers sarisfying the equarion* $x^2 - 4x + y^2 + 3 = 0$. *If the maximum and minimum values of* $x^2 + y^2$ *are* a *and* b *respectively, then the numerical value of* $a - b$ *is*

(a) 1 (b) 2

(c) 7 (d) 8

Ques. 18. *The domain of*

$$f(x) = cos^{-1}\left(\frac{6-3x}{4}\right) + cosec^{-1}\left(\frac{x-1}{2}\right) \text{ is}$$

(a) $[3, 10/3]$ (b) $[2/3, {}^{10}/_{3}]$

(c) $(-\infty, 1] \cup [3, \infty)$ (d) none of these

Ques. 19. $\lim_{x\to 0} \frac{ln(1+\{x\})}{\{x\}}$, *where* $\{.\}$ *denotes fractional part of* x, *is*

(a) 1 (b) ln 2

(c) does not exist (d) none on these

Ques. 20. $f(x) = \int_0^x (e^t - 1)(t-1)(sin\, t - cos\, t)\, sin\, t\, dt$, $x \in \left(-\frac{\pi}{2}, 2\pi\right)$

then $f(x)$ *is decreasing in*

(a) $(-\pi/2, 0) \cup (\pi/4, 1) \cup (\pi, 5\pi/2)$

(b) $(-\pi/2, \pi/4) \cup (1, \pi) \cup (5\pi/4, 2\pi)$

(c) $(\pi/4, 1) \cup (\pi,) \cup (\pi, 5\pi/4)$

(d) $(0, \pi/4,) \cup (1, \pi) \cup (5\pi/4, 2\pi)$