KAYSONS EDUCATION
ENLIGHTNING YOUR FUTURE

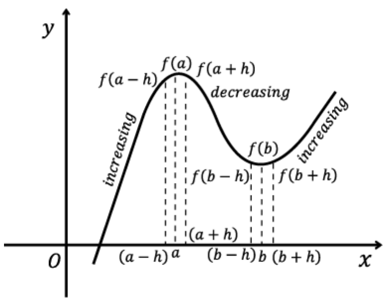
Chapter 8 Maxima Minima

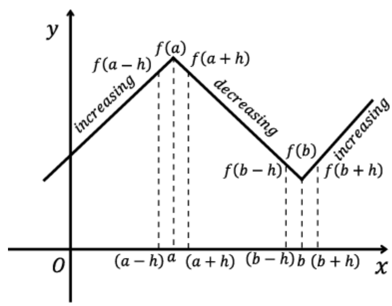
Day 1

Definition

A function $f\,(x)$ is said to have a maxima at $x = a$ if $f\,(a)$ is greatest of all values in the suitably small neighbourhood of 'a', where $x = a$ is an interior point in the domain of $f\,(x)$. Analytically this means $f(a) \geq f(a+h)$&$f(a) \geq f(a-h)$ where h is sufficiently small quantity.

Similarly, a function $y = f\,(x)$ is said to have a minimum at $x = b$ if $f\,(b)$ is smallest of all values in the suitably small neighborhood of 'b', where $x = b$ is an interior point in the domain of $f\,(x)$. Analytically this means $f(b) \leq f(b+h)$&$f(b) \leq f(b-h)$ where h is sufficiently small.

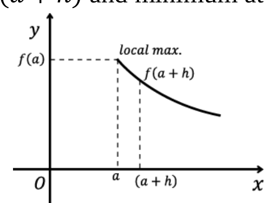


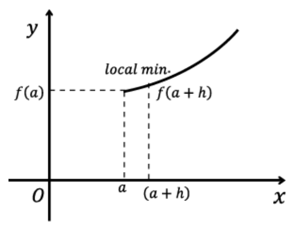


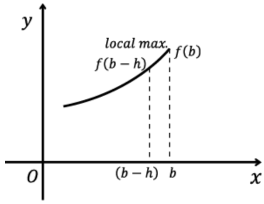
Maxima and Minima at end point

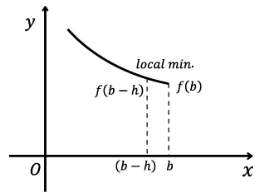
Definition

Let a function $f(x)$ is defined on [a,b]. Then f(x) has a local maximum at $x = a$(left end point) if f$(a) > f(a+h)$ and minimum at $x = a \; if \; f(a) < f(a+h)$.









Similarly, $f(x)$ has a local maximum at $x = b$(right end point) if $f(b) > f(b-h)$ and minimum at $x = b \; if \; f(a) < f(b-h)$.

<u>Method of finding extrema of continuous functions</u>

At points of extrema the derivative f'(x) either doesnot exits or if exist it is equal to zero. The points at which $f'(x)$=0 or doesnot exist are known as critical points.

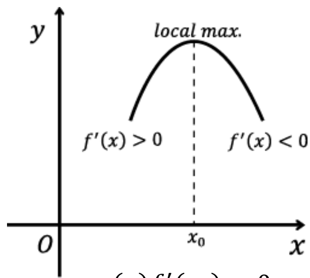
1st derivative test

The following test applies to a continuous function in order to get the extrema.

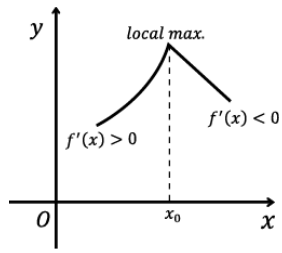
(a)At a critical point $x = x_0$

(i) If $f'(x)$ changes from positive to negative at x_0 while moving from left to right,

$$i.e.\begin{cases} f'(x) > 0; x < x_0 \text{ and} \\ \quad f'(x) < 0\,; x > x_0 \end{cases}$$



$(a) f'(x_0) = 0$

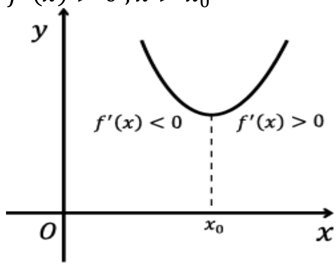


$(b) f'(x_0) does\ not\ exist$

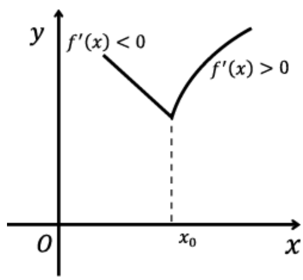
Then $f(x\ )$ has a local maximum value at $x = x_0$

(ii) If $f'(x)$ changes from negative to positive at x_0 while moving from left to right,

$$i.e.\begin{cases} f'(x) < 0; x < x_0 \; and \\ \quad f'(x) > 0\,; x > x_0 \end{cases}$$



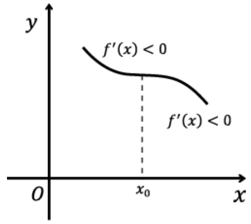
$(a) f'(x_0) = 0$

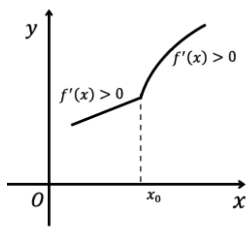


$(b) f'(x_0) = does\ not\ exist$

Then $f(x\ )$ has a local maximum value at $x = x_0$

(iii) If sign of $f'(x)$ does not change at x_0 then $f(x)$ has neither a maximum or minimum at x_0



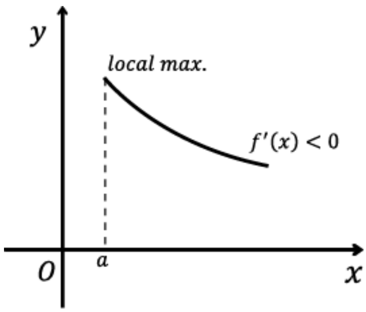


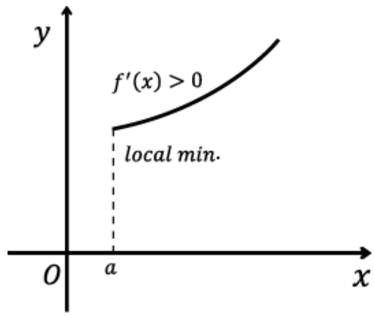
$(a) f'(x_0) = (inflection\ point)$ $(b) f'(x_0) = does\ not\ exist$

(b)At a left end point a

If $f(x)$ is defined on [a,b].

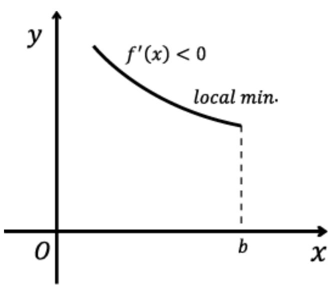
If $f'(x) < 0\ for\ x > a$, then $f(x)$ has a local maximum and If $f'(x) > 0\ for\ x > a$, then $f(x)$ has a local minimum at $x = a$

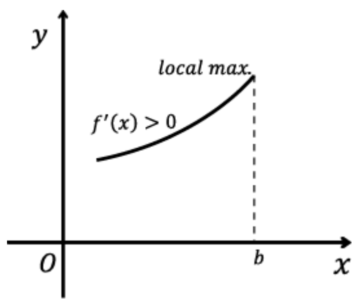




At a right end point b

If $f'(x) < 0\ for\ x < b$, then $f(x)$ has a local maximum and If $f'(x) > 0\ for\ x < b$, then $f(x)$ has a local maximum at $x = b$.





Remember that in a continuous function maximum and minimum values occur alternately i.e. between two successive maxima there is one minimum and between two successive minima there is one maximum.

Illustration

Find the local maxima or local minima, if any of the function

$f(x) = sin^4x + cos^4x, 0 < x < \pi/2$

Using the first derivative test.

Solution

We have,

$$y = f(x) = sin^4x + cos^4x$$

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$
$$\Rightarrow -4\cos x \sin x(\cos^2 x - \sin^2 x)$$
$$\Rightarrow -2\sin 2x \cos 2\,x$$
$$\Rightarrow -sin\ 4x$$

For a local maximum or a local minimum, we have

$$\frac{dy}{dx} = 0$$
$$\Rightarrow -\sin 4x = 0$$
$$\Rightarrow \sin 4\,x = 0$$
$$\Rightarrow 4x = \pi$$
$$\left[\because 0 < x < \frac{\pi}{2} \therefore 0 < 4x > 2\pi\right]$$
$$\Rightarrow x = \frac{\pi}{4}$$

Consider $x = \frac{\pi}{4}$

$$x < \frac{\pi}{4}$$
$$\Rightarrow 4x < \pi$$
$$\Rightarrow \sin 4x > 0$$
$$\Rightarrow -\sin 4x < 0$$
$$\Rightarrow \frac{dy}{dx} < 0$$

In the right nbd of

$$x = \frac{\pi}{4}$$
$$x > \frac{\pi}{4}$$
$$\Rightarrow 4x > \pi$$
$$\Rightarrow \sin 4x < 0$$
$$\Rightarrow -\sin 4x > 0$$
$$\Rightarrow \frac{dy}{dx} > 0$$

Thus $\mathrm{dy/dx}$ changes sign from negative to positive as x increases through $\frac{\pi}{4}$. so , $x = \frac{\pi}{4}$ is a pint of local minimum.

The local minimum values is

$$f\left(\frac{\pi}{4}\right) = \left(\sin\frac{\pi}{4}\right)^4 + \left(\cos\frac{\pi}{4}\right)^4 = \frac{1}{2}$$

Illustration

Find the maxima and minimum value of function,

$$f(x) = 3x^2 + 6x + 8, x\epsilon\ R$$

Solution

We have,

$$f(x) = 3x^2 + 6x + 8$$

$\Rightarrow 3(x^2 + 2x + 1) + 5$

$\Rightarrow 3(x + 1)^2 + 5$

$\because 3(x + 1)^2 \geq 0$ *for all* $x \in R$

$\Rightarrow 3(x + 1)^2 + 5 \geq 5$ *for all* $x \in R$

$\Rightarrow f(x) \geq 5$ for all $x \varepsilon R$

Thus, 5 is the minimum value of $f(x)$ which it attains at $x = -1$.

Since $f(x)$ can be made as large as we please, therefore the maximum value does not exist.

Question Practice Online

Method of 2nd derivative

It must be remembered that this method is not applicable to those critical points where $f'(x)$ remains undefined.

First we find the root of $f'(x) = 0$. Suppose x=a is one of the roots of $f'(x) = 0$.

Now find $f''(x)$ at $x = a$.

(i) If $f''(a)$ = negative; then $f(x)$ is maximum at $x = a$.

(ii) If $f''(a)$ = positive; then $f(a)$ is minimum at $x = a$.

(ii) If $f''(a)$ = zero;

Then we find $f''(x)\ at\ x = a$.

If $f''(a) \neq o$ then $f(x)$ has neither maximum nor minimum (inflexion point).

At $x = a$.

But if $f''(a) = 0$, then find $f^{iv}(a)$;

If $f^{iv}(a)$ = positive then $f(x)$ is minimum at $x = a$

If $f^{iv}(a)$ = negative then $f(x)$ is maximum at $x = a$

And so on, process is repeated till point is discussed.

Concept of Global Maximum / Minimum

Let $y = f(x)$ be a given function with domain D.

Let [a, b], then global maximum / minimum of $f(x)$ in $[a, b]$ is basically the greatest / least value of $f(x)$ in $[a, b]$.

Global maximum in $[a, b]$ would always occur at critical points of $f(x)$ with in $[a, b]$ or at the end points of the interval.

Global Maximum / Minimum in [*a*, *b*]

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, find out all critical points of $f(x)$ in $[a, b]$ (i.e., all points at which $f'(x) = 0$).

Let $c_1, c_2, c_3, \ldots \ldots \ldots \ldots, c_n$ be the points at which $f'(x) = 0$.

And let $f(c_1), f(c_2), \ldots \ldots \ldots, f(c_n)$ be the values of the function at these points.

Then $M_1 \rightarrow$ Global maximum or greatest value.

and $M_2 \rightarrow$ Global minimum or lest vlaue.

Where,

$$M_1 = max.\{f(a), f(c_1), f(c_2), \ldots \ldots \ldots, f(c_n), f(b)\}$$

And

$$M_2 = min.\{f(a), f(c_1), f(c_2), \ldots \ldots \ldots, f(c_n), f(b)\}$$

Then M_1 is the greatest value or global maximum in [a, b]

And M_2 is the least value or global minimum in [a, b]

Global Maximum / Minimum in (a, b)

Method for obtaining the greatest and least values of $f(x)$ in (a, b) is almost same as the method Used for obtaining the greatest and lest values in $[a, b]$,

However a caution may be taken;

Let

$$M_1 = max.\{f(a), f(c_1), f(c_2), \dots\dots., f(c_n),\}$$

And

$$M_2 = min.\{f(a), f(c_1), f(c_2), \dots\dots., f(c_n),\}$$

But if,

$$\lim_{x \to a^+} f(x) > M_1$$

Or

$$\lim_{x \to b^-} f(x) < M_2$$

$\Rightarrow f(x)$ would not posses global maximum or global minimum in (a, b).

This means the limiting values at the end points are greater than M_1 or less than M_2, then global maximum or global minimum does not exist in (a, b)