KAYSONS EDUCATION
ENLIGHTNING YOUR FUTURE

Chapter 7 Monotonicity

Day 1

Monotonic function

Monotonous behavior i.e. in the sense of ascending (increasing) or descending (decreasing). There are two types of monotonic function.

(i) Increasing function

It can be further studied under two subtopics

(i) Strictly increasing function-

A function f (x) is known as strictly increasing function in its domain if

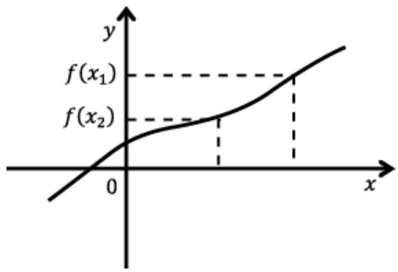
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ i.e. for the smaller inprt we have smaller output and for higher value of input we have higher output.

Nature of derivative of function

We know that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$x < (x+h) \ \{as\ h > 0\}$$



Hence,

$$f(x) < f(x+h)$$

$$f(x+h) - f(x) > 0$$

$$f'(x) = \frac{+ve}{-ve} = +ve$$

$$i.e. f'(x) > 0$$

$' f'(x)$ may also be zero but only at finite number of points and not in an interval.

(ii) Non decreasing function

A function f (x) is said to be non decreasing if for $x_1 < x_2$

It follows $f(x_1) \leq f(x_2)$

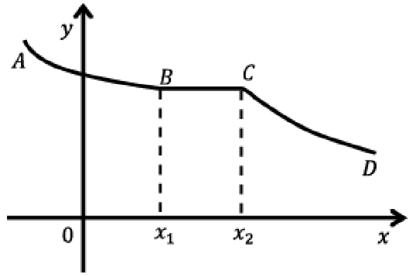
For AB & CD portions

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

And for BC, $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$

Hence, as a whole we can say that for non decreasing functions

$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.



Obviously, for this $f'(x) \geq 0$ where equality holds for horizontal path of the graph i.e. in the interval of BC.

(ii) Decreasing Function

(i)Strictly decreasing function

A function f (x) is said to be strictly decreasing in its domain if it follows $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ i.e. for smaller value of input we have higher output and for higher input we have smaller output.

Nature of derivative

We have,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

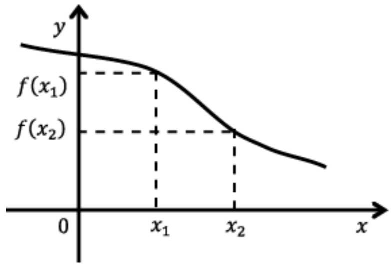
$$x < x + h \,\{as\ h > 0\}$$

$$f(x) > f(x+h)$$

$$\Rightarrow f(x+h) - f(x) < 0$$

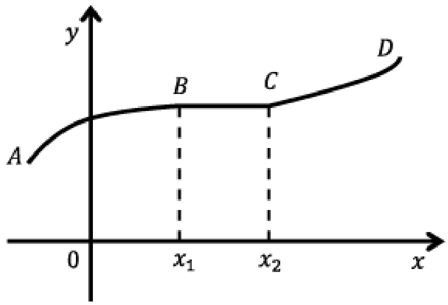
$$f'(x) = \frac{-ve}{+ve} = -ve.$$

$f'(x) < 0$ for strictly decreasing function. $f'(x)$ may also be zero but only at finite number of points.



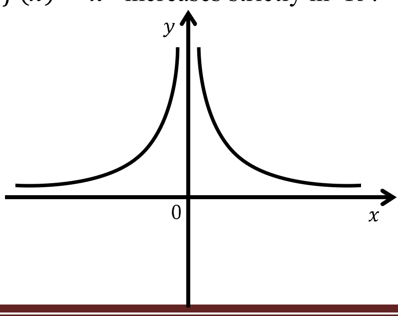
(ii) Non increasing functions

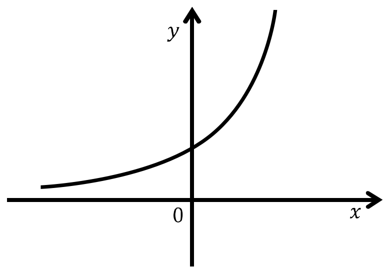
A function $f(x)$ is called non-increasing if for $x_1 < x_2 \Rightarrow f(x_1) \geq (x_2)$ For AB & CD portion $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ while $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$ for BC portion. It is clear that $f'(x) \leq 0$ for this case.



Illustration

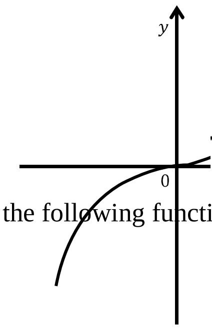
(i)$f(x) = x^3$ increases strictly in 'R'.



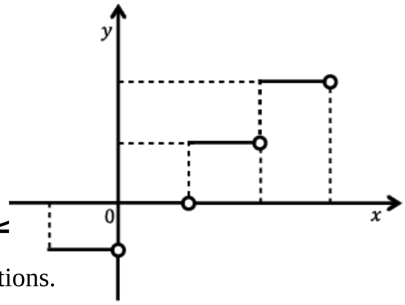


(ii)$f(x) = \frac{1}{|x|}$ is strictly increasing in $]-\infty, 0[$ and strictly decreasing in $]0, \infty[$.

(iii)$f(x) = e^x$ is strictly increasing in 'R'



(iv)$f(x) = [x]$ is increasing but not strictly increasing (i.e. non decreasing) in 'R'.



Illustration

Find the interval of increase or decrease of the following functions.

$(a) f(x) = 2x^3 + 3x^2 - 12x + 1$

Solution

(a)

$f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2)$

When $(x^2 + x - 2) = 0, x = -2, 1$

Since $f'(x) < 0$ in $]-2, 1[$

Hence the function decreases in the interval $]-2, 1[$ and increases when $x < -2$ or $x > 1$ $\{f'(x) > 0$ in these intervals$\}$.

Illustration

Find the interval in which $f(x) = x^3 - 3x^2 - 9x + 20$ is strictly increasing or strictly decreasing

Solution

Given,

$$f(x) = x^3 - 3x^2 - 9x + 20$$
$$\Rightarrow f'(x) = 3x^2 - 6x - 9$$
$$\Rightarrow f'(x) = 3(x^2 - 2x - 3)$$
$$\Rightarrow f'(x) = 3(x - 3)(x + 1)$$

Using number line method,

$$\Rightarrow f'(x) > 0 \; for \; x \; \epsilon(-\infty, -1) \cup (3, \infty)$$
$$f'(x) < 0 \;\; for \; x \; \epsilon \; (-1,3)$$

Thus, f(x) is strictly increasing for $x \; \epsilon(-\infty, -1) \cup (3, \infty)$ and strictly decreasing for $x \; \epsilon \; (-1,3)$.

Illustration

Find the interval in which $f(x) = (x+1)^3(x-3)^3$ is increasing or decreasing.

Solution

We have

$$f(x) = (x+1)^3(x-3)^3$$
$$f'(x) \Rightarrow 3(x+1)^2(x-3)^3 + 3(x+1)^3(x-3)^2$$
$$\Rightarrow 3(x+1)^2(x-3)^2(x+1+x-3)$$
$$\Rightarrow 6(x+1)^2(x-3)^2(x-1)$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 \Rightarrow 6(x+1)^2(x-3)^2(x-1) > 0$$
$$\Rightarrow x-1 > 0$$
$$[\because 6(x+1)^2(x-3)^2 > 0]$$
$$\Rightarrow x > 1$$
$$\Rightarrow x \epsilon (1, \infty)$$

So, $f(x)$ is increasing on $(1, \infty)$

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 \Rightarrow 6(x+1)^2(x-3)^2(x-1) < 0$$
$$\Rightarrow x-1 < 0$$
$$[\because 6(x+1)^2(x-3)^2 > 0]$$
$$\Rightarrow x < 1$$
$$\Rightarrow x \epsilon (-\infty, 1)$$

So, $f(x)$ is decreasing on $(-\infty, 1)$

Illustration

Find the interval in which the function

$f(x) = \ln(1+x) - \frac{2x}{2+x}$ is increasing or decereasing.

Solution

We have

$$f(x) = \ln(1+x) - \frac{2x}{2+x}$$
$$\Rightarrow f'(x) = \frac{1}{1+x}\frac{d}{dx}(x+1) - \frac{(2+x).2-2x(0+1)}{(2+x)^2}$$
$$\Rightarrow \frac{1}{1+x} - \frac{4}{(2+x)^2}$$
$$\Rightarrow \frac{(2+x)^2-4(1+x)}{(2+x)^2(1+x)} = \frac{x^2}{(2+x)^2(1+x)}$$
$$\Rightarrow \left(\frac{x}{2+x}\right)^2\left(\frac{1}{x+1}\right)$$

For $f(x)$ to be increasing, we must have

$$f'(x) > 0 = \left(\frac{x}{2+x}\right)^2\left(\frac{1}{x+1}\right) > 0$$
$$\Rightarrow \frac{1}{x+1} > 0$$
$$\left[\because \left(\frac{x}{2+x}\right)^2 > 0\right]$$
$$\Rightarrow x + 1 > 0$$
$$\Rightarrow x > -1$$
$$\Rightarrow x \in (-1, \infty)$$

So, $f(x)$ is increasing on $(-1, \infty)$

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0 = \left(\frac{x}{2+x}\right)^2\left(\frac{1}{x+1}\right) < 0$$
$$\Rightarrow \frac{1}{x+1} < 0$$
$$\left[\because \left(\frac{x}{2+x}\right)^2 > 0\right]$$
$$\Rightarrow x + 1 < 0$$
$$\Rightarrow x < -1$$

but ln $(1 + x)$ is defind function $x < -1$.

Hence function is only increasing function.

Illustration

Determine the value of x for which

$f(x) = \frac{x-2}{x+1}$, $x \neq -1$ is increasing or decreasing .

Solution

We have,

$$f(x) = \frac{x-2}{x+1}, x \neq -1$$
$$\Rightarrow f'(x) = \frac{(x+1)1-(x-2).1}{(x+1)^2}$$
$$\Rightarrow \frac{3}{(x+1)^2}, x \neq -1$$

Clearly,

$$f'(x) = \frac{3}{(x+1)^2} > 0 \text{ for all } x \in R \text{, except } x = -1$$

So, $f(x)$ is increasing on R – {–1}

Illustration

Separate $[0, \pi/2]$ into subintervals in which $f(x) = \sin 3x$ is increasing or decreasing.

Solution

We have

$$f(x) = \sin 3x \;\Rightarrow f'(x) = 3\cos 3x$$

Now,

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < 3x < 3\pi/2$$

Since cosine function is positive in first quadrant and negative in the second and third quadrants. Therefore, we consider the following cases.

Case I

When

$$0 < 3x < \pi/2$$
$$0 < 3x < \frac{\pi}{2} \Rightarrow \cos 3x > 0$$
$$\Rightarrow 3\cos 3x > 0 \Rightarrow f'(x) > 0$$

Thus,

$$f'(x) > 0 for\ 0 < 3x < \pi/2 i.e. 0 < x < \pi/6$$

So, $f(x)$ is increasing on $(0, \pi/6)$

Case II

When

$$\pi/2 < 3x < 3\pi/2$$
$$\pi/2 < 3x < \frac{3\pi}{2} \Rightarrow \cos 3x < 0$$
$$\Rightarrow 3\cos 3x < 0 \Rightarrow f'(x) < 0$$

Thus,

$$f'(x) < 0 \text{ for } \pi/2 < 3x < 3\pi/2 i.e. \pi/6 < x < \pi/2$$

So, $f(x)$ is decreasing on $(\pi/6, \pi/2)$

Hence, $f(x)$ is increasing on $(0, \pi/6)$ and decreasing on $(\pi/6, \pi/2)$.

Illustration

Find the intervals in which

$f(x) = \frac{x}{\ln x}$ is increasing or decreasing.

Solution

Note the domain of $f(x)$ is the set of all positive real numbers i.e. $f(x)$ is defined for all $x \neq 0$.

Now,

$$f(x) = \frac{x}{\ln x}$$
$$\Rightarrow \frac{\ln x - 1}{(\ln x)^2}$$

For f(x) to be increasing, we must have

$$f'(x) > 0$$
$$\Rightarrow \frac{\ln x - 1}{(\ln x)^2} > 0$$
$$\Rightarrow \ln x - 1 > 0$$
$$[\because \ln_a x > N \Rightarrow x > a^N for\ x > 0]$$
$$\Rightarrow \ln x > 1$$
$$\Rightarrow x > e^1$$

$$\begin{bmatrix} \because \ln_a x > N \Rightarrow x > a^N for\ a > 1 \\ Here, e > 1\ \therefore \ln_e x > 1 \Rightarrow x > e^1 \end{bmatrix}$$

$$\Rightarrow x \in (e, \infty).$$

So, $f(x)$ is increasing on (e, ∞)

For $f(x)$ to be decreasing, we must have

$$f'(x) < 0$$

$$\Rightarrow \frac{\ln x - 1}{(\ln x)^2} < 0$$

$$\Rightarrow \ln x - 1 < 0$$

$$[\because (\ln x)^2 > 0\ for \Rightarrow x > 0]$$

$$\Rightarrow \ln x < 1$$

$$\Rightarrow x < e^1$$

$$[\because f(x)\ is\ defined\ for\ x > 0]$$

$$\Rightarrow x \in (0, e) - \{1\}$$

So, $f(x)$ is decreasing on $(0, e) - \{1\}$

Illustration

Separate the interval [0, π/2] into subintervals in which $f(x) = \sin^4 x + \cos^4 x$ is increasing or decreasing.

Solution

We have

$$f(x) = \sin^4 x + \cos^4 x$$

$$\Rightarrow f'(x) = 4\sin^3 x \cos 4 - 4\cos^3 x \sin x$$

$$\Rightarrow f'(x) = -4\sin x \cos x(\cos_2 x - \sin^2 x)$$

$$\Rightarrow f'(x) = -2(2\sin x \cos x)(\cos 2x)$$

$$\Rightarrow f'(x) = -2\sin 2x \cos 2\ x$$

$$\Rightarrow f'(x) = -\sin 4x$$

Now,

$$0 < x < \frac{\pi}{2} \Rightarrow 0 < 4x < 2\pi$$

Since sine function is positive in first and second quadrant and negative in the third and fourth quadrants. we consider the following cases.

Case I

When

$$0 < 4x < \pi \Rightarrow \sin\ 4x > 0$$

$$\Rightarrow -\sin 4x < 0$$

$$\Rightarrow f'(x) < 0$$

$$\because f'(x) < 0 for\ 0 < 4x < \pi\ i.e. 0 < x < \pi/4$$

So, $f(x)$ is decreasing on (0, π/4)