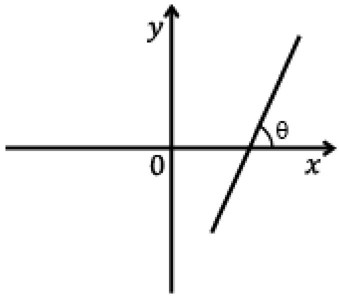
Chapter 6 Tangents and Normals

Day 1

Basic definition

Slope (Gradient) of a line

The trigonometrical tangent of the angle that a line makes with the positive direction of x-axis in anticlockwise sense is called the slope or gradient of the line .



Slope of a line in terms of coordinates of any two points on it

Let P(x_1, y_1) and Q(x_2, y_2) be two points on a line. Then its slope m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

Slope of a line when its equation is given

The slope of a line whose equation is $ax+by+c=0$ is given by

$$m = -\frac{a}{b} = -\frac{\text{coeff.of } x}{\text{coeff.of } y}$$

Angle between two lines

The angle θ between two lines having slopes m_1 and m_2 is given by

$$\tan\theta = \pm\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

When two line are parallel, their slopes are equal.

$$m_1 = m_2$$

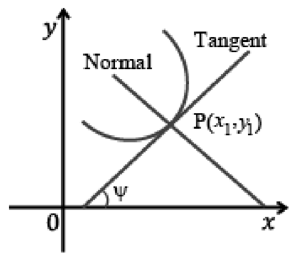
Two lines of slopes m_1 and m_2 are perpendicular, then

$$m_1 . m_2 = -1$$

Equation of a straight line

The equation of a straight line passing through a point (x_1, y_1) and slope m is

$$y - y_1 = m(x - x_1)$$



$$\text{slope of tangent} = \left(\frac{dy}{dx}\right)_P = \tan\theta$$

$$\text{slope of normal} = -\frac{1}{\left(\frac{dy}{dx}\right)_P}$$

Illustration

Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$

Solution

The equation of the given curve is

$$y^2 = 4ax \dots\dots\dots (i)$$

Differentiating (i) with respect to x, we get

$$2y\frac{dy}{dx} = 4a$$
$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(at^2,2at)} = \frac{2a}{2at}$$
$$\Rightarrow \frac{1}{t}$$

So, that equation of the tangent at $(at^2,2at)$ is

$$y - 2at = \left(\frac{dy}{dx}\right)_{(at^2,2at)}(x - at^2)$$
$$\text{or } y - 2at = \frac{1}{t}(x - at^2)$$
$$\text{or } ty = x + at^2$$

And, the equation of the normal at $(at^2, 2at)$ is

$$y - 2at = -\frac{1}{\left(\frac{dy}{dx}\right)_{(at^2,2at)}}(x - at^2)$$
$$\Rightarrow\ y - 2at = -\frac{1}{\frac{1}{t}}(x - at^2)$$
$$\Rightarrow\ y - 2at = -t(x - at^2)$$
$$\Rightarrow y + tx = 2at + at^3$$

Illustration

Find the equation of the tangent and normal to $16x^2 + 9y^2 = 144$ at the point (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$

Solution

The equation of the given curve is

$$16x^2 + 9y^2 = 144 \dots\dots\dots\dots (i)$$

Since (x_1, y_1) lies on (i), therefore

$$16x_1^2 + 9y_1^2 = 144$$
$$\Rightarrow 16(2)^2 + 9y_1^2 = 144$$
$$\Rightarrow y_1^2 = \frac{80}{9}$$
$$\Rightarrow y_1 = \frac{4\sqrt{5}}{3}$$

$[\because y_1 > 0]$

So, the given point is

$$\left(2, \frac{4\sqrt{5}}{3}\right)$$

Now,

$$16x^2 + 9y^2 = 144$$
$$\Rightarrow 32x + 18y\frac{dy}{dx} = 0$$

[Differentiating with respect to x.]

$$\Rightarrow \frac{dy}{dx} = -\frac{16x}{9y}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)} = -\frac{16\times 2}{9\times\frac{4\sqrt{5}}{3}}$$
$$\Rightarrow -\frac{8}{3\sqrt{5}}$$

So, the equation of the tangent at

$$\left(2, \frac{4\sqrt{5}}{3}\right) \ is$$
$$y - \frac{4\sqrt{5}}{3} = \left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)}(x-2)$$
$$\Rightarrow y - \frac{4\sqrt{5}}{3} = -\frac{8}{3\sqrt{5}}(x-2)$$
$$\Rightarrow 8x + 3\sqrt{5}y - 36 = 0$$

And, the equation of the normal at

$$\left(2, \frac{4\sqrt{5}}{3}\right) \ is$$
$$y - \frac{4\sqrt{5}}{3} = -\frac{1}{\left(\frac{dy}{dx}\right)_{\left(2, \frac{4\sqrt{5}}{3}\right)}}(x-2)$$
$$\Rightarrow y - \frac{4\sqrt{5}}{3} = \frac{-1}{-\frac{8}{3\sqrt{5}}}(x-2)$$
$$\Rightarrow y - \frac{4\sqrt{5}}{3} = \frac{3\sqrt{5}}{8}(x-2)$$
$$\Rightarrow 9\sqrt{5}\,x - 24y + 14\sqrt{5} = 0$$

Illustration

Find the equation of the tangent line to y = $2x^2 + 7$ which is parallel to the line $4x - y + 3 = 0$.

Solution

Let the point of contact of the required tangent line be (x_1, y_1)

The equation of the given curve is $y = 2x^2 + 7$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 4x$$
$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 4x_1$$

Since the line $4x - y + 3 = 0$ is parallel to the tangent at (x_1, y_1).

$$\therefore \text{slope of the tangent at } (x_1, y_1) = \text{slope of the line } 4x - y + 3 = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{-4}{-1}$$

$$\left[\because \text{slope} = -\frac{\text{coeff.of } x}{\text{coeff.of } y}\right]$$

$$\Rightarrow 4x_1 = 4$$

$$\Rightarrow x_1 = 1$$

Now,

(x_1, y_1) lies on

$$y = 2x^2 + 7.$$

$$\therefore y_1 = 2x_1^2 + 7$$

$$\Rightarrow y_1 = 2 + 7$$

$$\Rightarrow 9$$

$$[\because x_1 = 1]$$

So, the coordinates of the point of contact are (1, 9). Hence, the required equation of the tangent line is

$$y - 9 = 4(x - 1)$$

$$\Rightarrow 4x - y + 5 = 0$$

Illustration

Find the equation of tangent to the curve $y = x^3 + 2x + 6$ which is perpendicular to the line $x + 14y + 4 = 0$.

Solution

Let the coordinates of the point of contact be (x_1, y_1).

Then

$$y_1 = x_1^3 + 2x_1 + 6 \ldots\ldots (i)$$

$$[\because (x_1, y_1) \text{ lies on line } y = x^3 + 2x + 6]$$

The equation of the curve is

$$y = x^3 + 2x + 6$$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = 3x^2 + 2$$

$$\Rightarrow \left(\frac{dy}{dxs}\right)_{(x_1,y_1)} = 3x_1^2 + 2$$

Since the tangent at (x_1, y_1) is perpendicular to the line $x + 14y + 4 = 0$,

Therefore,

$$\text{slope of the tangent at } (x_1, y_1) \times \text{slope of the line} = -1$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,y_1)} \times -\frac{1}{14} = -1$$

$$\Rightarrow (3x_1^2 + 2)\left(-\frac{1}{14}\right) = -1$$

$$\Rightarrow 3x_1^2 + 2 = 14$$

$$\Rightarrow x_1 = \pm 2$$

When

$$x_1 = 2, y_1 = 2^3 + 2 \times 2 + 6 = 18$$

When

$$x_1 = -2, y_1 = (-2)^3 + 2 \times (-2) + 6 = -6$$

So, the coordinates of the points of contact are (2, 18) and (-2, -6)

The equation of the tangent at (2, 18) is

$$y - 18 = 14(x - 2)$$
$$\Rightarrow 14x - y - 10 = 0$$
$$\left[\text{using } y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1,y_1)} (x - x_1)\right]$$

The equation of the tangent at (-2, -6) is

$$y— 6 = 14(x— 2)$$
$$\Rightarrow 14x - y + 22 = 0$$

Illustration

Show that the curve $4x = y^2$ and $4xy = k$ cut at right angles, if $k^2 = 512$.

Solution

Curve are $4x = y^2$ …………….(i) and $4xy = k$ ……………………….(ii)

Eliminating y we get,

$$4x = \left(\frac{k}{4x}\right)^2$$
$$\Rightarrow 64x^3 = k^2$$
$$\Rightarrow x = \frac{k^{2/3}}{4}$$

Put in (i)

$$y = k^{1/3},$$

Point of intersection is

$$\left(\frac{k^{2/3}}{4}, k^{1/3}\right)$$

Differentiating (i) with respect to x, we get

$$4 = 2y\frac{dy}{dx}$$
$$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$$
$$\Rightarrow \left.\frac{dy}{dx}\right]_{\left(\frac{k^{2/3}}{4}, k^{1/3}\right)} = \frac{2}{k^{1/3}} \dots\dots\dots\dots..(iii)$$

Differentiating (ii) with respect to x, we get

$$4\left(x\frac{dy}{dx} + y\,.1\right) = 0$$
$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$
$$\Rightarrow \left.\frac{dy}{dx}\right]_{\left(\frac{k^{2/3}}{4}, k^{1/3}\right)} = -\frac{k^{1/3} \times 4}{k^{2/3}}$$