Chapter 5 Application and Derivatives

Day 1

Derivative as a rate measurer

Derivative as the time rate of change

If a variable quantity u is some function of time $u = f(t)$, then a small change in time Δt will have a corresponding change Δu in u. Thus, the average rate of change is $\Delta u/\Delta t$. When limit $\Delta t \to 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to time at the instant t

$i.e. \lim\limits_{\Delta t \to 0} \frac{\Delta u}{\Delta t} = \frac{du}{dt}$ Hence, it is clear that the rate of change of any variable quantity is its derivative with

respect to time.

The d. c. of y with respect to x i.e. dy/dx is nothing but the rate of increase of y relative to x.

Related rates

In many practical problems, it has been found that several variables are related by an expression. The time rate of change of certain variables are known and the time rates of change of others are to be found. In such cases first of all a relation between the variables is established. Then we differentiable the relation with respect to time. On differentiation we get a resulting equation containing known and unknown rates. The value of known rates are substituted and thus we obtain unknown rate.

Illustration

If the radius of a circle be increasing at a uniform rate of 2 cm/sec, find the rate of increase of area, at the instant when the radius is 20 cm.

Solution

Given

$$\frac{dr}{dt} = 2\ cm/sec$$

$$(r \to radius)$$

Now, area of a circle is given by $A = \pi r^2$

Differentiating w. r. to time t we get

$$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi(20)2 = 80\ cm^2/sec$$

Illustration

If r be the radius S the surface and V the volume of a spherical bubble, prove that

(i)

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(ii)

$$\frac{dS}{dt} \propto r$$

Solution

(i) Since $V = \frac{4}{3}\pi r^3$

$\therefore \frac{dV}{dt} = \frac{4}{3}\pi.3r^2\frac{dr}{dt} = 4\pi.r^2\frac{dr}{dt}$

(ii)$S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi.r\frac{dr}{dt}$

Thus,

$\frac{dV}{dS} = \frac{dV/dt}{dS/dt} = \frac{1}{2}.r$

$\frac{dV}{dS} \propto r$

Illustration

Water is dripping out from a conical funnel of semi-vertical angle π/4 at the uniform rate of 2 cm^2/sec in its surface area through a tiny hole at the vertex in the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

Solution

Let VAB be a conical funnel of semi-vertical angle π/4.At any time t the water in the cone also forms a cone. Let r be its radius, l be the slant height and S be the surface area.

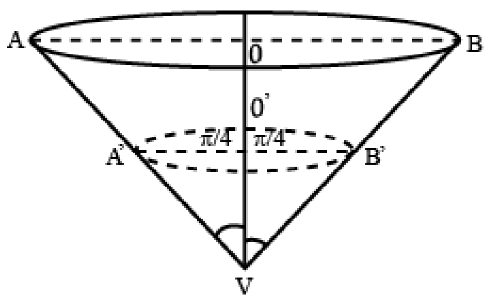
Then,

$VA' = l, O'A' = r \; and \; \angle A'VO' = \pi/4.$

In $\Delta VO'A'$, we have

$\cos\frac{\pi}{4} = \frac{VO'}{VA'} = \frac{VO'}{l} \; and \; \sin\frac{\pi}{4} = \frac{O'A'}{VA'} = \frac{O'A'}{l}.$

$\therefore VO' = l\cos\frac{\pi}{4} \; and \; O'A' = l\sin\frac{\pi}{4}.$



The surface area S is given by

$$S = \pi(O'A')(VA')$$

$[using\ S = \pi rl]$

$$S = \pi l sin\frac{\pi}{4}.l = \pi l^2\sin\frac{\pi}{4} = \frac{\pi l^2}{\sqrt{2}} \Rightarrow \frac{dS}{dt} = \frac{2\pi l}{\sqrt{2}}\frac{dl}{dt}$$

It is given that

$$\frac{dS}{dt} = 2\,cm^2/sec.$$

$$\therefore 2 = \frac{2\pi l}{\sqrt{2}}\frac{dl}{dt} \Rightarrow \frac{dl}{dt} = \frac{\sqrt{2}}{\pi\, l}$$

Putting l = 4, we get

$$\frac{dl}{dt} = \frac{\sqrt{2}}{4\,\pi} cm/sec.$$

Thus, the rate of decrease of the slant height is $\sqrt{2}/4\pi$ cm/sec.

Illustration

A man 1.6m high walks at the rate of 30metres per minute away from a lamp which is 4m above the ground. How fast is the man's shadow lengthening.

Solution

Let PQ = 4m be the height of the pole and AB = 1.6m be the height of the man.

Let the end of the shadow is R and it is a distance of l form A when the man is at a distance x from

PQ at some instant.

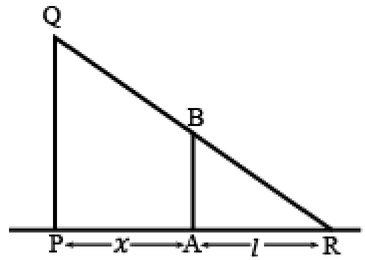
Since ΔPQR & ΔABR are similar

We have,

$\frac{PQ}{AB} = \frac{PR}{AR}$

$\Rightarrow \frac{4}{1.6} = \frac{x+l}{l} = \frac{x}{l} + 1$

$\Rightarrow \frac{x}{l} = \frac{2}{3} \Rightarrow l = \frac{2}{3}\, x$



$$\therefore \frac{dl}{dt} = \frac{2}{3}\frac{dx}{dt} = \frac{2}{3}.30$$

$$\Rightarrow \frac{dl}{dt} = 20\, m/min.$$

Illustration

A man is moving away from a tower 41.6m high at the rate of 2m/sec. Find the rate at which the angle of elevation of the top of tower is changing, when he is at a distance of 30m from the foot of the tower. Assume that the eye level of the man is 1.6m from the ground.

Solution

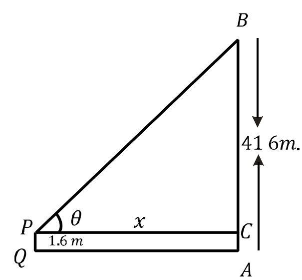
Let AB be the tower. Let at any time t, the man be at a distance of x meters from the tower AB and let θ be the angle of elevation at that time.

Then,

$\tan\theta = \frac{BC}{PC} \Rightarrow \tan\theta = \frac{40}{x}$

$\Rightarrow x = 40\cot\theta \qquad \ldots.(i)$

$\Rightarrow \frac{dx}{dt} = -40\, cosec^2\theta\, \frac{d\theta}{dt}$



We are given that dx/dt = 2m/sec. Therefore

$$2 = -40\, cosec^2\theta \frac{d\theta}{dt}$$

$$\Rightarrow \frac{d\theta}{dt} = -\frac{1}{20\, cosec^2\theta} \qquad \dots (ii)$$

When x = 30, from (i) we have

$$\cot\theta = \frac{30}{40} = \frac{3}{4}$$

So,

$$cosec^2\theta = 1 + \cot^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$$

Substituting $cosec^2\,\theta = 25/16$ in (ii), we get

$$\frac{d\theta}{dt} = -\frac{1}{20\times\frac{25}{16}} = -\frac{4}{125} radians/sec$$

Thus, the angle of elevation of the top of tower is changing at the rate of 4/125 radians/sec.

Question Practice Online

Differentials, Errors and Approximations

Remark-1

$$\Delta y = \frac{dy}{dx}.\Delta x$$

Remark-2

Let $y = f(x)$ be a function of x, and let Δx be a small change in x. let Δy be the corresponding change in y. then

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$$
$$\Rightarrow \frac{\Delta y}{\Delta x} = f'(x) + \varepsilon, \text{where } \varepsilon \to 0 \; as \; \Delta x \to 0$$
$$\Rightarrow \; \Delta y = f'(x)\Delta x + \varepsilon\,\Delta x \Rightarrow \; \Delta y = f'(x)\Delta x, \text{approximately}$$

or,

$$\Delta y = \frac{dy}{dx}\,\Delta x, \text{approximately}$$
$$\left[\because f'(x) = \frac{dy}{dx}\right]$$

Some important terms

Absolute Error- the error Δx in x is called the absolute error in x.

Relative Error- If Δx is an error in x, then $\Delta x/x$ is called the relative error in x.

Percentage Error-If Δx is an error in x, then $\Delta x/x \times 100$ is called percentage error in x.

Let $y = f(x)$ be a function of x, and let Δx be a small change in x. let the corresponding change in y be Δy. Then

$$y + \Delta y = f(x + \Delta x)$$

But,

$$\Delta y = \frac{dy}{dx}.\Delta x = f'(x)\Delta x, approximately$$
$$\therefore f(x + \Delta x) = y + \Delta y$$
$$\Rightarrow y + f'(x).\Delta x, approximately$$
$$\Rightarrow y + \frac{dy}{dx}.\Delta x, approximately$$

Illustration

If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the approximate change in y ?

Solution

Let $x = 2, x + \Delta x = 1.99$. then $\Delta x = 1.99 - 2 = -0.01$.

Let $dx = \Delta x = -0.01$

We have,

$$y = x^4 - 10$$
$$\therefore \frac{dy}{dx} = 4x^3 \Rightarrow \left(\frac{dy}{dx}\right)_{x=2} = 4(2)^3 = 32$$

Now,

$$dy = \frac{dy}{dx}.dx \Rightarrow dy = 32(-0.01) = -0.32$$

We know that Δy is approximately equal to dy.

$$\Delta y = -0.32 \text{ approximately}$$

So, approximate change in $y = -0.32$.
When $x = 2, y = 2^4 - 10 = 6$.
So, changed value of

$$y = y + \Delta y = 6 + (-0.32) = 5.68 \,.$$

Illustration

Find the percentage error in calculating the volume of a cubical box if an error of 1% is made in measuring the length of edges of the cube.

Solution

Let x be the length of an edge of the cube and y be its volume. Then $y = x^3$. Let Δx be the error in x and Δy be the corresponding error in y. Then

$$\frac{\Delta x}{x} \times 100 = 1 \text{ (given)} \qquad \dots (i)$$

And we have to find

$$\frac{\Delta y}{y} = 100.$$

Let $\Delta x = dx$. Then, form (i)

$$\frac{dx}{x} \times 100 = 1 \qquad\qquad \dots (ii)$$

Now,

$$y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2$$

$$dy = \frac{dy}{dx}.dx \Rightarrow dy = 3x^2.dx \Rightarrow \frac{dy}{y} = \frac{3x^2}{y}\ dx$$

$$\Rightarrow \frac{dy}{y} = \frac{3x^2}{x^3}\ dx$$

$$[\because y = x^3]$$

$$\Rightarrow \frac{dy}{y} = 3\frac{dx}{x} \Rightarrow \frac{dy}{y} \times 100 = 3\left(\frac{dx}{x} \times 100\right)$$

$$\Rightarrow \frac{dy}{y} \times 100 = 3(1) = 3$$

Since Δy is approximately equal to dy. Therefore,

$$\frac{dy}{y} \times 100 = 3 \Rightarrow \frac{\Delta y}{y} \times 100 = 3$$

So, there is 3% error in calculating the volume of the cube.

Illustration

Use differentials to find the approximate value of $\sqrt{0.037}$.

Solution

Let $y = f(x) = \sqrt{x}, x = 0.040$ and $x + \Delta x = 0.037$.
Then

$$\Delta x = 0.037 - 0.040 = -0.003 \,.$$

For x = 0.040, $y = \sqrt{0.040} = 0.2$

[putting $x = 0.040$ in $y = \sqrt{x}$]

Let $dx = \Delta x = 0.003$.

Now,

$$y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Rightarrow \left(\frac{dy}{dx}\right)_{x=0.040} = \frac{1}{2\sqrt{0.040}} = \frac{1}{0.4}$$

$$dy = \frac{dy}{dx}.dx \Rightarrow \frac{1}{0.4}(-0.003) = -\frac{3}{400}$$

Now, Δy is approximately equal to dy. So

$$\Delta y = -\frac{3}{400}.$$

Hence,

$$\sqrt{0.037} = y + \Delta y = 0.2 - \frac{3}{400} = 0.2 - 0.0075 = 0.1925\ .$$

Illustration

Using differentials find the approximate value of $\tan 46^0$, if is being given that $1^0 = 0.01745$ radians.

Solution

Let $y = f(x) = \tan x\, , x = 45^0 = (\pi/4)^c$ and $x + \Delta x = 46^0$.

Then $\Delta x = 1^0 = 0.01745$ radians.

For

$$x = \pi/4\, , y = f(\pi/4) = \tan \pi/4 = 1$$

Let $dx = \Delta x = 0.01745$

Now,

$$y = \tan x \Rightarrow \frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)_{x=\pi/4} = \sec^2 \pi/4 = 2$$

$$dy = \frac{dy}{dx}.dx \Rightarrow dy = 2(0.01745) = 0.03490\ .$$

But Δy is approximately equal to dy. So,

$$\Delta y = dy = 0.03490\ .$$

Hence,

$$\tan 46^0 = y + \Delta y = 1 + 0.03490 = 1.03490\ .$$

Question Practice Online

Rolle's Theorem

Let f be a real valued function defined on the closed interval [a, b] such that,

(i). f(x) is continuous in the closed interval [a, b],

(ii). f(x) is differentiable in the open interval $]a, b[$ and

(iii). f(a) = f(b)

Then there is at least one value of c of x in open interval $]a, b[$ for which $f'(c) = 0$

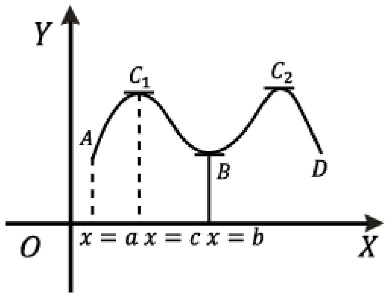
Geometrical proof

Consider the portion AB of the curve $y = f(x)$, lying between x = a and x = b, such that

(i). it goes continuously from A to B

(ii). it has tangent at every point between A and B and

(iii). ordinate of A = ordinate of B



From figure, it is clear that f(x) increases in the interval AC_1, which implies that $f'(x) > 0$ in this region and decreases in the interval C1B which implies $f'(x) < 0$ in this region. Now, since there is unique tangent to ne drawn on the curve lying in between A and B ad since each of them has a unique slope i.e., unique value of $f'(x)$,

Due to continuity and differentiability of the function f(x) in the region A to B.

There is a point is x = c where $f'(c) = 0$ should be zero.

Hence, $f'(c) = 0$ where a < c < b.

The following results are very helpful in doing so.

(i) A polynomial function is every where continuous and differentiable.

(ii) The exponential function, sine and cosine functions are everywhere continuous and differentiable.

(iii) Logarithmic functions is continuous and differentiable in its domain.

(iv) tan x is not continuous and differentiable at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots \dots \dots$

(v) |x| is not differentiable at x = 0.

(vi) if $f'(x)$ tends to $\pm\infty$ as $x \to k$, then f(x) is not differentiable at x = k.

Illustration

Verify Rolle's theorem for the function $f(x) = x^3 - 3x^2 + 2x$ in the interval [0, 2],

Solution

Here we observe that

(a) f(x) is polynomial and since polynomial are always continuous, f(x) is continuous in the interval [0, 2]

(b) $f'(x) = 3x^2 - 6x + 2$ clearly exists for all $x \in (0, 2)$. so f(x) is differentiable for all $x \in (0, 2)$ and

(c)

$$f(0) = 0, f(2) = 2^3 - 3.(2)^2 + 2(2) = 0$$
$$f(0) = f(2)$$

Thus, all the condition of Rolle's theorem are satisfied.

So, there must exists some $c \in (0, 2)$ such that $f'(c) = 0$

$$\Rightarrow f'(c) = 3c^2 - 6c + 2 = 0$$
$$\Rightarrow c = 1 \pm \frac{1}{\sqrt{3}}$$

Where $c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$ thus Rolle's theorem is verified.

Illustration

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals:

(i)

$$f(x) = 3 + (x-2)^{\frac{2}{3}}\ on\ [1, 3]$$

(ii)

$$f(x) = \tan x \quad on \quad [0, \pi]$$

Solution

(i)

We have,

$$f(x) = 3 + (x-2)^{2/3}, x \in [2, 3]$$
$$\therefore f'(x) = (2/3)(x-2)^{1/3}$$

Clearly, $\lim_{x \to 2} f'(x) = \infty$. So, f(x) is not differentiable at $x = 2 \in (1, 3)$.

Hence, Rolle's theorem is not applicable to $f(x) = 3 + (x-2)^{2/3}$ on the interval [1, 3] .

(ii)

We have

$$f(x) = \tan x, x \in [0, \pi].$$

Since $(\pi/2) \in [0, \pi]$ and f(x) is not continuous at $x = \pi/2$.

So, the condition of continuity at each point of $[0, \pi]$ is not satisfied.

Hence, Rolle's theorem is not applicable to $f(x) = \tan x$ on the interval [0, π].

Illustration

Verify Rolle's theorem for the function $f(x) = (x-a)^m.(x-b)^n$ on the interval [a, b], where m, n are positive integers.

Solution

We have $f(x) = (x-a)^m(x-b)^n$, where m, $n \in N$ On expanding $(x-a)^m$ and $(x-b)^n$ by binomial theorem and then taking the product, we find that f(x) is a polynomial of degree (m + n). since a polynomial function is every where differentiable and so continuous also. therefore,

(i) f(x) is continuous on [a, b]

(ii) f(x) is derivable on (a, b).

Also, $f(a) = f(b) = 0$.

Thus, all the three conditions of Rolle's theorem are satisfied. Now we have to show that there exists $c \in (a, b)$ such that $f'(c) = 0$

We have

$$f(x) = (x-a)^m (x-b)^n$$
$$\Rightarrow f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m\, n(x-b)^{n-1}$$
$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{m(x-b) + n(x-a)\}$$
$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{x(m+n) - (mb+na)\}$$
$$f'(x) = 0$$
$$\Rightarrow (x-a)^{m-1}(x-b)^{n-1}\{x(m+n) - (mb+na)\} = 0$$
$$\Rightarrow (x-a) = 0 \; or \; (x-b) = 0 \; or \; x(m+n) - (mb+na) = 0$$
$$\Rightarrow x = a \; or \; b \; or \; x = \frac{mb+na}{m+n}$$

Since $x = mb + na/m + n$ divides (a, b) into the ratio m:n, therefore $mb + na/m + n \in (a, b)$.

Thus, $c = mb + na/m + n \in (a, b)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

Illustration

Verify Rolle's theorem for each of the following functions on the indicated intervals:

$$f(x) = e^x(\sin x - \cos x) \; on \; [\pi/4, 5\pi/4].$$

Solution

Since an exponential function an sine and cosine functions are everywhere continuous and differentiable, therefore f(x) is continuous on $[\pi/4, 5\pi/4]$ and differentiable on $(\pi/4, 5\pi/4)$.

Also,

$$f\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$
$$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left(\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = e^{5\pi/4}\left(-\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right)$$
$$\Rightarrow \; e^{5\pi/4}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$
$$f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$$

Thus, f(x) satisfies all the three conditions of Rolle's theorem on $\pi/4, 5\pi/4$. Consequently there exists $c \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$.

Now,

$$f(x) = e^x(\sin x - \cos x)$$
$$\Rightarrow f'(x) = e^x(\sin x - \cos x) + e^x(\cos x + \sin x) = 2e^x \sin x$$
$$\Rightarrow f'(x) = 0 \Rightarrow 2e^x \sin x = 0 \Rightarrow \sin x = 0$$
$$[e^x \neq 0]$$
$$\Rightarrow x = \pi$$

Thus, $c = \pi \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$.

Hence, Rolle's theorem is verified.

Question Practice Online

Lagrange's Theorem

Statement- let f(x) be a function defined on [a, b] such that

(i) it is continuous on [a, b],

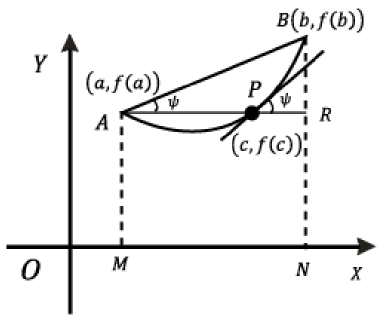
(ii) it is differentiable on (a, b).

Then there exists a real number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Geometrical interpretation

Let f(x) be a function defined on [a, b], and let APB be the curve represented by $y = f(x)$. then coordinates of *A* and *B* are *(a, f(a))* and *(b, f(b))* respectively. Suppose the chord AB makes an angle ψ with the axis of *x*. Then from the triangle ARB, we have



$\tan\Psi = \frac{BR}{AR} \Rightarrow \tan\Psi = \frac{f(b)-f(a)}{b-a}$

By Lagrange's mean value theorem, we have $f'(c) = \frac{f(b)-f(a)}{b-a}$ $\therefore \tan\Psi = f'(c)$

Slope of the chord AB = slope of the tangent at (c, f(c))

Illustration

Verify Lagrange's mean value theorem for the function $f(x) = (x-3)(x-6)(x-9)$ on the interval [3, 5].

Solution

We have,

$$f(x) = (x-3)(x-6)(x-9) = x^3 - 18x^2 + 99x - 162.$$

Since a polynomial function is every where continuous and differentiable, therefore f(x) is continuous on [3,5] and differentiable on (3,5).

Thus, both the condition of Lagrange' mean value theorem are satisfied. So, there must exist at least one real number $c \epsilon (3,5)$ such that

$$f'(c) = \frac{f(5)-f(3)}{5-3}$$

Now,

$$f(x) = x^3 - 18x^2 + 99x - 162$$
$$\Rightarrow f'(x) = 3x^2 - 36x + 99$$
$$f(5) = (5-3)(5-9) = 8$$

And

$$f(3) = 0$$
$$\therefore f'(x) = \frac{f(5)-f(3)}{5-3}$$

$$\Rightarrow 3x^2 - 36x + 99 = \frac{8-0}{5-3}$$
$$\Rightarrow 3x^2 - 36x + 99 = 4$$
$$\Rightarrow 3x^2 - 36x + 95 = 0$$
$$\Rightarrow x = \frac{36 \pm \sqrt{1296-1140}}{6} = \frac{36 \pm 12.49}{6}$$
$$\Rightarrow 8.8, 4.8$$

Thus

$$c = 4.8 \in (3,5)$$

Such that

$$f'(c) = \frac{f(5)-f(3)}{5-3}$$

Hence, Lagrange's mean value theorem is verified.

Illustration

Using Lagrange's mean value theorem, find a point on the curve $y = \sqrt{x-2}$ defined on the interval [2, 3], where the tangent is parallel to the chord joining the end points of the curve.

Solution

Let $f(x) = \sqrt{x-2}$ .since for each $x \in [2,3]$, the function *f*(*x*) attains a unique definite value. So, *f*(*x*) is continuous on [2,3].

Also,

$$f'(x) = \frac{1}{2\sqrt{x-2}} \text{ exists for all } x \in (2,3). \text{So, } f(x) \text{ is differentiable on } (2,3).$$

Thus, both the condition of Lagrange's mean value theorem are satisfied.

Consequently there must exist some $c \in (2,3)$ such that

$$f'(c) = \frac{f(3)-f(2)}{3-2}$$

Now,

$$f(x) = \sqrt{x-2}$$
$$\Rightarrow f'(x) = \frac{1}{2\sqrt{x-2}}$$
$$\Rightarrow f(3) = 1 \text{ and } f(2) = 0$$
$$\Rightarrow \therefore f'(x) = \frac{f(3)-f(2)}{3-2}$$
$$\Rightarrow \frac{1}{2\sqrt{x-2}} \Rightarrow \frac{1-0}{3-2}$$
$$\Rightarrow \frac{1}{2\sqrt{x-2}} \Rightarrow 1$$
$$\Rightarrow 4(x-2) = 1$$
$$\Rightarrow x - 2 = \frac{1}{4}$$
$$\Rightarrow x = \frac{9}{4}$$

Thus,

$$c = \frac{9}{4} \in (2,3) \text{ such that } f'(c) = \frac{f(3)-f(2)}{3-2}$$

Now,

$$f(c) = \sqrt{\frac{9}{4} - 2} = \frac{1}{2}$$

Thus, (c,f(c)) i.e.(9/4,1/2) is a point on the curve

$$y = \sqrt{x - 2}$$

Such that the tangent at it is parallel to the chord joining the end points of the curve.

Illustration

Verify Lagrange's mean value theorem for the functions on the indicated intervals.

$$f(x) = 2\sin x + \sin 2x \text{ on } [0, \pi]$$

Solution

Since sin x and sin 2x are everywhere continuous and differentiable, therefore f(x) is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$. Thus, f(x) satisfies both the conditions of Lagrange's mean value theorem. Consequently,

There exists at least one $c\,\epsilon(0, \pi)$ such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

Now,

$$f(x) = 2\sin x + \sin 2x$$
$$\Rightarrow f'(x) = 2\cos x + 2\cos 2x$$
$$\Rightarrow f(0) = 0$$

And

$$f(\pi) = 2\sin \pi + \sin 2\pi = 0$$
$$\therefore f'(x) = \frac{f(\pi) - f(0)}{\pi - 0}$$
$$\Rightarrow 2\cos x + 2\cos 2x = \frac{0 - 0}{\pi - 0}$$
$$\Rightarrow 2\cos x + 2\cos 2x = 0$$
$$\Rightarrow \cos x + \cos 2x = 0$$
$$\Rightarrow \cos 2x = -\cos x$$
$$\Rightarrow \cos 2x = \cos(\pi - x)$$
$$\Rightarrow 2x = \pi - x$$
$$\Rightarrow 3x = \pi$$
$$\Rightarrow x = \frac{\pi}{3}$$

Thus,

$$c = \frac{\pi}{3}\,\epsilon\,(0, \pi)$$

Such that

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

Hence, Lagrange's mean value theorem is verified.

Question Practice Online