Chapter 4 Differentiation

Day 1

Introduction

Consider a function $y = f(x)$

Now if x is incremented by a value Δx then the value of y also changes by Δy.

$$\therefore y + \Delta y = f(x + \Delta x)$$

Thus the change in value of y, is,

$$\Delta y = (y + \Delta y) - y$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

The derivative of function $y = f(x)$ at a point (x, y) is the slope of the tangent of the function at that point.

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Hence the derivative of the function $y = f(x)$ is found by the above method

The fundamental method to find derivatives of a function $y = f(x)$with respect to x is called the First principle to find derivatives of a function

Note:

This method is also known as:

1. differentiation by definition
2. differentiation y abinitio

Illustration

Differentiation of cos x from first principles

Solution

Let

$$f(x) = \cos x$$

Then

$$f(x + h) = \cos(x + h)$$

$$\frac{d}{dx}\big(f(x)\big) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{\cos(x+h) - cosx}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{-2\sin\left(\frac{2x+h}{2}\right) sin\left(\frac{h}{2}\right)}{h}$$

$$\left[\because \cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right).\sin\left(\frac{C-D}{2}\right)\right]$$

$$\Rightarrow -\lim_{h\to 0} \sin(x+h/2).\lim_{h\to 0} \sin\frac{(h/2)}{(h/2)} = (-sinx).1$$

$$\because \left[\lim_{h\to 0} \frac{\sin(h/2)}{(h/2)} = 1\right]$$

$$\Rightarrow -sinx$$

Illustration

Differentiation of cot x from first principles

Solution

Let

$$f(x) = \cot x$$

Then

$$f(x+h) = \cot(x+h)$$

$$\frac{d}{dx}\big(f(x)\big) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\cot(x+h)-cotx}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\frac{\cos(x+h)}{\sin(x+h)} - \frac{cosx}{sinx}}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{\sin x\cos(x+h) - \cos x\sin(x+h)}{hsin\, x\sin(x+h)}$$

$$\Rightarrow \lim_{h\to 0} \frac{\sin\big(x-(x+h)\big)}{h\sin x\sin(x+h)}$$

$$[\because \sin A\cos B - \cos A\sin B = \sin(A-B)]$$

$$\Rightarrow -\lim_{h\to 0} \frac{\sin h}{h}.\lim_{h\to 0} \frac{1}{\sin x\sin(x+h)}$$

$$[\because \sin(-h) = sinh]$$

$$\Rightarrow (-1)\frac{1}{\sin x\sin x} = -cosec^2 x$$

$$\left[\because \lim_{h\to 0} \frac{sinh}{h} = 1\right]$$

$$\frac{d}{dx}\big(f(x)\big) = -cosec^2 x, \frac{d}{dx}(\cot x) = -cosec^2 x.$$

Illustration

$$x\tan^{-1} x$$

Solution

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$where\ y = f(x) = \tan^{-1} x$$

$$\Rightarrow \lim_{h\to 0} \frac{(x+h)\tan^{-1}(x+h) - x\tan^{-1} x}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{x\left(\tan^{-1}(x+h)-\tan^{-1}x\right)+h\tan^{-1}(x+h)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{x}{h}\tan^{-1}\left(\frac{(x+h)-x}{1+(h+x)x}\right)+\lim_{h\to 0}\frac{h\tan^{-1}(x+h)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{x}{1+x^2+hx}.\lim_{h\to 0}\left(\tan^{-1}\frac{\left(\frac{h}{1+x^2+hx}\right)}{\frac{h}{(1+x^2+hx)}}\right)+\tan^{-1}x$$

$$\Rightarrow \frac{x}{1+x^2}.1+\tan^{-1}x$$

$$\left[\because \lim_{x\to 0}\frac{\tan^{-1}x}{x}=1\right]$$

$$\Rightarrow \tan^{-1}x+\frac{x}{1+x^2}$$

Illustration

$\cos(logx)$

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\cos\big(log(x+h)\big)-\cos(logx)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{2\sin\frac{(log(x+h)+logx)}{2}.\sin\left(\frac{logx-\log(x+h)}{2}\right)}{h}$$

$$\Rightarrow \lim_{h\to 0}\sin\left(\frac{\log(x+h)x}{2}\right).\frac{\sin\left(1/2\,log\left(\frac{x}{x+h}\right)\right)}{h/2}$$

$$\Rightarrow \sin\left(\frac{logx^2}{2}\right)\lim_{h\to 0}\frac{\sin\left[-1/2\log\left(\frac{x+h}{x}\right)\right]}{h/2}$$

$$\Rightarrow \sin logx\lim_{h\to 0}\frac{\sin\left[-1/2\left(h/x-\frac{h}{2x^2}+\cdots\right)\right]}{h/2}$$

$$\Rightarrow -\sin logx\lim_{h\to 0}\frac{\sin h/2\left(\frac{1}{x}-\frac{h}{2x^2}+\cdots\right)}{h/2}$$

$$\Rightarrow -\sin logx\lim_{h\to 0}\frac{\sin h/2(1/x-h/2x^2+\cdots)}{h/2\,(1/x-h/2x^2+\cdots)}\lim_{h\to 0}(1/x-h/2x^2+\cdots)$$

$$\Rightarrow -\sin\log x\,.1\left(\frac{1}{x}\right)$$

$$\Rightarrow -\frac{\sin(logx)}{x}$$

Illustration

Find the derivative of $\sqrt[3]{\sin x}$ by first priciple

Solution

$$f(x) = \sqrt[3]{\sin x}$$

$$\Rightarrow \frac{d}{dx}\big(f(x)\big) = \lim_{h\to 0}\frac{f(x+h)+f(x)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\sqrt[3]{\sin(x+h)}-\sqrt[3]{\sin x}}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left\{\sqrt[3]{\sin(x+h)}\right\}^3-\left\{\sqrt[3]{\sin x}\right\}^3}{h\left[\sin^{\frac{2}{3}}(x+h)+\sin^{\frac{2}{3}}x+\sin^{\frac{1}{3}}(x+h)\sin^{\frac{1}{3}}x\right]}$$

$$\left[\because a-b=\frac{a^3-b^3}{a^2+ab+b^2}\right]$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin(x+h)-sinx}{h}\times\frac{1}{\sin^{2/3}(x+h)+\sin^{2/3}x+\sin^{1/3}(x+h)}$$

$$\sin^{1/3}x$$

$$\Rightarrow \lim_{h\to 0}\frac{\sin(h/2)\cos(x+h/2)}{h/2}\lim_{h\to 0}\frac{1}{\sin^{2/3}(x+h)+\sin^{2/3}x+\sin^{1/3}(x+h)\sin^{1/3}x}$$

$$\Rightarrow \cos x.\frac{1}{\sin^{2/3}x+\sin^{2/3}x+\sin^{1/3}x.\sin^{1/3}x}$$

$$\Rightarrow \frac{cosx}{3\sin^{2/3}x}$$

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