Chapter 3 Continuity and Differentiability

Day 1

Continuity of a function

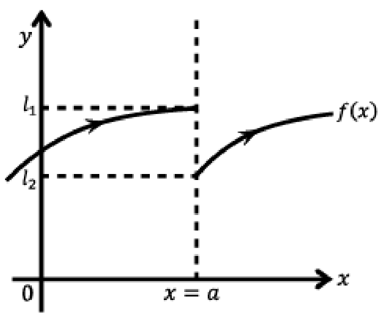
A function f(x) is a said to be continuous at x=a; where a$\in$domain of f(x)

If

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

Graphical View

(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists but are not equal.



Here,

$$\lim_{x\to a^-} f(x) = l_1$$
$$\lim_{x\to a^+} f(x) = l_2$$

$\therefore$ $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists but are not equal.

Thus, f(x) is discontinuous at x = a.

It does not matter whether f(a) exists or not.

Illustration

If $f(x) = \begin{cases} 2x+3, when\ x<0 \\ 0, when\ x=0. \\ x^2+3, when\ x>0 \end{cases}$ $Discuss\ the\ continuity.$

Solution

Here,

$$f(x) = \begin{cases} 2x+3, when\ x<0 \\ 0, when\ x=0 \\ x^2+3, when\ x>0 \end{cases}$$

$\therefore$ RHL at x = 0, let x = 0 + h

i.e.,

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\{(0+h)^2+3\} = 3$$
$$\Rightarrow \lim_{x\to 0^+} f(x) = 3$$

Again, LHL at x = 0,

Let x = 0-h

i.e.,

$$\lim_{x\to0^-} f(x) = \lim_{h\to0} f(0-h) = \lim_{h\to0}\{2(0-h)+3\} = 3$$
$$\Rightarrow \lim_{x\to0^-} f(x) = 3$$

But f(0) = 0

Therefore,

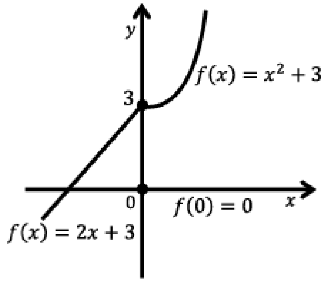
$$\lim_{x\to0^+} f(x) = \lim_{x\to0^-} f(x) = 3 \neq f(0)$$

Thus, f(x) is discontinuous at $x \to 0$

Graphically:-

Here,

$$\lim_{x\to0^-} f(x) = 3$$
$$\lim_{x\to0^+} f(x) = 3$$
$$f(x) = 0$$



Thus,

$$\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x) = 3 \neq f(0)$$

Hence, f(x) is discontinuous at x = 0

Illustration

$If\ f(x) = \frac{x^2-1}{x-1}$. Discuss the continuity $at\ x \to 1$.

Solution

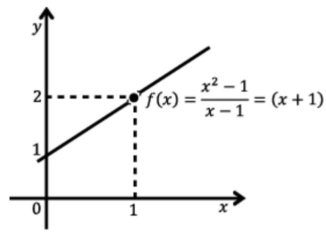
Here,

$$f(x) = \frac{x^2-1}{x-1}$$
$$\lim_{x\to1} f(x) = \lim_{x\to1}\frac{x^2-1}{x-1} = \lim_{x\to1}\frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to1}(x+1) = 2$$

But f(1) = 0/0 (in determined form)

$\therefore$ f(1) is not defined at x = 1

Hence, f(x) is discontinuous at x = 1



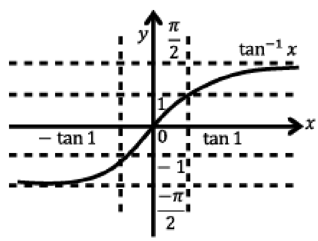
Graphically

Which shows,

$$\lim_{x\to1} f(x) = 2$$

But f(1) is not defined.

So, f(x) is discontinuous at x = 1.



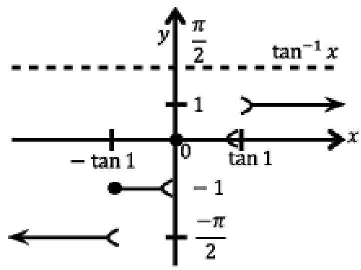
Illustration

Discuss the continuity of $f(x) = [\tan^{-1} x]$.

Solution

We know $y = \tan^{-1} x$ could be plotted as;

Thus, $f(x) = [\tan^{-1} x]$ could b plotted as;



Which clearly represents graph is breaked at
{- tan 1, 0, tan 1}
∴ f(x) is not continuous when $x \in \{-\tan 1, 0, \tan 1\}$

Continuity at end points

Let a function $y = f(x)$ is defined on [a, b].
Then the function f(x) is said to be continuous at the left end x = a if

$$f(a) = \lim_{x \to a^+} f(x)$$

If f(x) is said to be continuous at the right end x = b if,

$$f(b) = \lim_{x \to b^-} f(x)$$

Kinds of Discontinuity

Let the point x = a be the limit point in the domain of definition of y = f(x).
Discontinuity of 1st kind: In this kind of discontinuity the RHL and LHL of the function $y = f(x)$ are existent (i.e. are finite and definite) at x = a and if

(i) $\lim_{h \to 0} f(a-h) = \lim_{h \to 0} f(a+h) \neq f(a)$ Then f(x) is said to have first kind removable discontinuity. This kind of discontinuity can be removed by putting $f(a) = \lim_{x \to a} f(x)$.

(ii) $\lim_{h \to 0} f(a-h) \neq \lim_{h \to 0} f(a+h)$ Then f(x) is said to have find kind non-removable discontinuity.

The value $\lim_{h \to 0} f(a+h) - \lim_{h \to 0} f(a-h)$ is called jump discontinuity of f(x) at x = a.

Discontinuity of 2nd kind: If at least one of $\lim_{h \to 0} f(a+h)$ and $\lim_{h \to 0} f(a-h)$ is non –existent or infinite then f(x) is said to have discontinuity of 2nd kind at x = a.

Illustration

Show the function,

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, when\ x \neq 0 \\ 0, when\ x = 0 \end{cases}$$

has non-removable discontinuity at x = 0.

Solution

We have,

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, when\ x \neq 0 \\ 0, when\ x = 0 \end{cases}$$

∴ RHL at x = 0, let x = 0 + h

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0 + h) = \lim_{h \to 0} \frac{e^{\frac{1}{0+h}} - 1}{e^{\frac{1}{0+h}} + 1} = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}}$$

$$\Rightarrow \lim_{x \to 0^+} f(x) = \frac{1 - 0}{1 + 0} = 1 \left[as\ h \to 0; \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty; \frac{1}{e^{1/h}} \to 0\right]$$

$$\therefore \lim_{x \to 0^+} f(x) = 1$$

Again, LHL at x = 0, let x = 0-h

$$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0 - h) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = \frac{0 - 1}{0 + 1} = -1$$

$$\Rightarrow \lim_{x \to 0^-} f(x) = -1$$

$$\Rightarrow \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x).$$

Thus, f(x) has non-removable discontinuity.

Illustration

$$Show\ f(x) = \frac{1}{|x|}$$

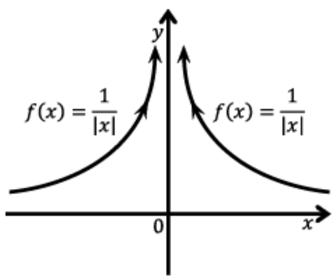
has discontinuity of second kind at x = 0.

Solution

Here,

$$f(x) = \frac{1}{|x|} = \infty$$

Which shows function has discontinuity of second kind.



Graphically

Here, the graph is broken at x = 0 as $x \to 0 \Rightarrow f(x) \to \infty$

Therefore, f(x) has discontinuity of second kind.

Jump discontinuity

A function f(x) is said to have a jump discontinuity at a point x = a if,

$$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$$

And f(a) may be equal to either of previous limits.

Illustration

$f(x) = [x]$; [ ] denotes greatest integer has jump discontinuity at all integer values.

Properties of continuity function

Theorem (i): If the functions f(x) and g(x) are continuous at a point x = a then the sum $\emptyset = f(x) + g(x)$ is also continuous at that point x = a.

Proof : Since f(x) and g(x) are continuous at a point x = a we can write

$$\lim_{x\to a} f(x) = f(a) \textit{ and } \lim_{x\to a} g(x) = g(a)$$

Now

$$\lim_{x\to a} \emptyset(x) = \lim_{x\to a}\{f(x) + g(x)\}$$

$$\Rightarrow \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$(\textit{using theorems on limit})$$

$$\Rightarrow f(a) + g(a) = \emptyset(a)$$

Thus, the function $\emptyset(x) = f(x) + g(x)$ is continuous.

Similarly, we can prove the following theorems

Theorem (ii): The product to two continuous functions is a continuous function.

Theorem (iii): The ratio of two continuous functions is a continuous function, provided the denominator does not vanish at the point under consideration.

Theorem (iv): If u = g(x) is continuous at x = a and f(u) is continuous at the point u0=g(a), then the composite function f{g(x)} is continuous at the point x = a.

Illustration

Discuss the continuity of the function,

$$f(x) = \lim_{n\to\infty} \frac{\ln(2+x) - x^{2n}\sin x}{1 + x^{2n}} \textit{ at } x = 1.$$

Solution

We have,

$$f(1) = \lim_{n\to\infty} \frac{\ln 3 - \sin 1}{2} = \frac{1}{2}(\ln 3 - \sin 1) \qquad \dots.(i)$$

We know that,

$$\lim_{n\to\infty} x^{2n} = \begin{cases} 0, if\ x^2 < 1 \\ \infty, if\ x^2 > 1 \end{cases}$$

$\therefore$ for $x^2 < 1$, we have

$$f(x) = \lim_{n\to\infty} \frac{\ln(2+x) - x^{2n}\sin x}{1+x^{2n}} \to \ln(2+x) = \ln 3$$

Again for $x^2 > 1$, we have

$$f(x) = \lim_{n\to\infty} \frac{x^{1/2n}\ln(2+x) - \sin x}{1+\frac{1}{x^{2n}}} \to -\sin(x)$$

Here, as $x \to 1$

$$\lim_{x\to 1^-} f(x) = \ln(3)\ \ and\ \lim_{x\to 1^+} f(x) = -\sin 1$$

So,

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x).$$

Therefore, f(x) is not continuous at x = 1.

Illustration

Let

$$f(x) = \begin{cases} \{1+|\sin x|\}^{\frac{a}{|\sin x|}};\ -\pi/6 < x < 0 \\ b;\quad x = 0 \\ e^{\frac{tan2x}{\tan 3x}};\ 0 < x < \pi/6 \end{cases}$$

Determine a and b such that f(x) is continuous at x = 0.

Solution

Since f is continuous at x = 0.

Therefore, RHL = LHL = f(0)

RHL at x = 0

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} e^{\tan 2h/\tan 3h}$$

$$\Rightarrow \lim_{h\to 0} e^{\frac{\tan 2h}{2h}\cdot\frac{3h}{\tan 3h}\cdot\frac{2}{3}} = e^{2/3} \qquad \ldots.(i)$$

Again LHL at x = 0:

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0}\{1+|\sin(0-h)|\}^{a/|\sin(0-h)|}$$

$$\Rightarrow \lim_{h\to 0}\{1+|\sin h|\}^{a/|\sin h|}$$

$$\Rightarrow e^{\lim_{h\to 0}|\sin h|\cdot\frac{a}{|\sin h|}} = e^{a} \qquad \ldots.(ii)$$

And

$$f(0) = b. \qquad \ldots.(iii)$$

Thus,

$$e^{2/3} = e^{a} = b \Rightarrow a = 2/3\ \ and\ b = e^{2/3}$$

Illustration

Find the points of discontinuity of

$$y = \frac{1}{u^2+u-2}$$

$$where\ u = \frac{1}{x-1}$$

Solution

The function $u = f(x) = \frac{1}{x-1}$ is discontinuous at the point x = 1. …(i)

The function

$$y = g(x) = \frac{1}{u^2 + u - 2} = \frac{1}{(u+2)(u-1)}$$

is discontinuous at $u = -2$ and $u = 1$.

When

$$u = -2, \frac{1}{x-1} = u = -2$$

$$\Rightarrow x - 1 = -\frac{1}{2} \Rightarrow x = \frac{1}{2} \qquad \dots (ii)$$

When

$$u = 1, \frac{1}{x-1} = u = 1$$

$$\Rightarrow x - 1 = 1$$

$$\Rightarrow x = 2 \qquad \dots (iii)$$

Hence, the composite function $y = g\big(f(x)\big)$ is discontinuous at three points $x = 1/2, 1, 2$.

A list of continuous functions

Function f(x)	Interval in which f(x) is continuous
1. $constant\ c$	$(-\infty, \infty)$
2. $x^n, n\ is\ an\ integer \geq 0$	$(-\infty, \infty)$
3. $x^{-n}, n\ is\ a\ positive\ integer$	$(-\infty, \infty) - \{0\}$
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a^n$	$(-\infty, \infty)$
6. $\frac{p(x)}{q(x)}, where\ p(x)\ and\ q(x) are\ polynomial\ in\ x$	$(-\infty, \infty) - \{x: q(x) = 0\}$
7. $\sin x$	$(-\infty, \infty)$
8. $\cos x$	$(-\infty, \infty)$
9. $\tan x$	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
10. $\cot x$	$(-\infty, \infty) - \{n\pi: n \in I\}$
11. $\sec x$	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
12. $cosec\ x$	$(-\infty, \infty) - \{n\pi: n \in I\}$
13. e^x	$(-\infty, \infty)$
14. $\log_e x$	$(0, \infty)0$

Differentiability

Before introducing the concept and condition of differentiability, it is important to know differentiation and the conce

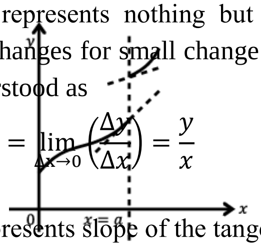
Differential coefficient of a function y=f(x) is written as

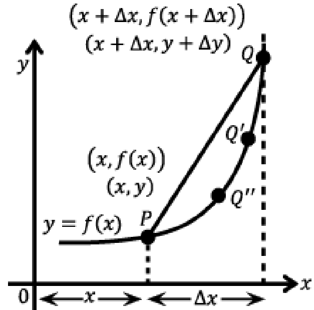
$\frac{d}{dx}[f(x)]$ *or* $f'(x)$ *or* $f^{(1)}(x)$ *as defined by*

$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

f'(x) represents nothing but ratio by which *f*(x) changes for small change is x and can be understood as

$f'(x) = \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x}\right) = \frac{y}{x}$





Then *f'*(x) represents slope of the tangent drawn at point 'x' of the curve *f*(x) .

Let us understand the geometrical meaning of differentiation:

Slope of PQ

$$PQ = \frac{y_Q - y_P}{x_Q - x_P}$$

$$\Rightarrow \frac{(y + \Delta y) - y}{(x + \Delta x) - x} \; or \; \frac{f(x + \Delta x) - f(x)}{(x + \Delta x) - x}$$

$$\Rightarrow \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Let point Q approach point P , along the curve =y=f(x) i.e. $\Delta x \to 0$. Then, we observe graphically that the slope of chord PQ becomes the slope of the tangent at the point P, which is written as

$$\frac{dy}{dx} \; or \; f'(x^+)$$

Since, point Q is approaching point P from the Right Hand side, we obtain *f'*(x) as follows:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(Right hand derivative)

Similarly,

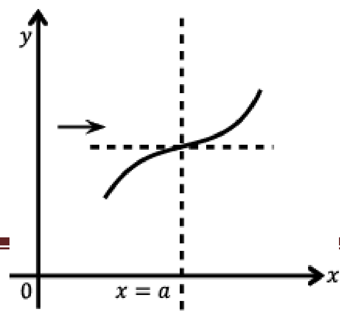
$$f'(x^-) = \lim_{\Delta x \to 0} \frac{f(x - \Delta x) - f(x)}{-\Delta x} \; , \Delta x > 0$$

(Left hand derivative)

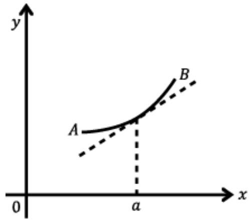
Note: For a function to be differentiable at x=a, we should have $f'(a^-) = f'(a^+)$ i.e.

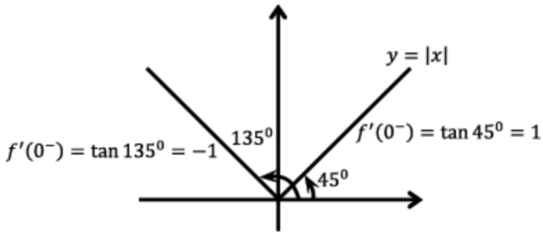
$$\lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

From the above graphs m one must not infer that a curve is non-differentiable only at points discontinuity. Non differentiability conditions also arise when the curve is continuous and the curve suddenly changes direction. The easiest



example of a curve being continuous and non-differentiable is
$y = |x|\ at\ x = 0$. However when there is a smooth change or gradual change in slope or trajectory of curve the derivative exists.





Few more illustrations are given below:
Refer to the following graphs:

Fig. (I)

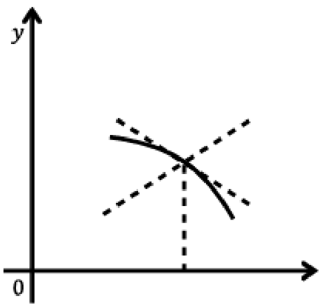


Fig (II)

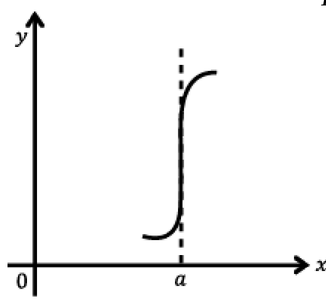


Fig (III)

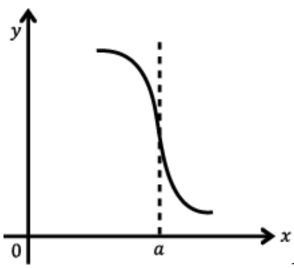


Fig (VI)

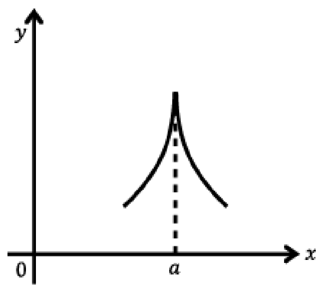


Fig (V)

In figure (i), f'(a) exists and is finite. In figure (ii) both $f'(a^-)\ and\ f'(a^+)$ exist but they are not equal. Hence f(a) does not exist. Figure (iii) and (iv) have infinite derivatives, i.e.
$f'(a) = +\propto\ and\ f'(a) = -\propto$ respectivly. In case of figure (v) we have
$f'(a^-) = +\propto\ and\ f'(a^+) = -\propto$ f'(a) does not exist.

Illustration

Let [.] denotes the greatest integer function and $f(x) = [\tan^2 x]$, then

$(a) \lim_{x\to 0} f(x)\ does\ not\ exists$ $(b) f(x) is\ continuous\ at\ x = 0$

$(c)\ f(x)\ is\ not\ differentiable\ at\ x = 0$ $(d) f'(0) = 1$

Solution

Here [.] denotes the greatest integral function, thus

As

$$-45^o < x < 45^o$$
$$\Rightarrow \tan(-45^o) < \tan x < \tan 45^0$$
$$\Rightarrow -1 < \tan x < 1$$
$$\Rightarrow 0 < \tan^2 x < 1$$

Hence

$$f(x) = [\tan^2 x] = 0$$

Hence, $f(x)$is zero for all values of x from $(-45^o)$to (45^o).Thus, $f(x)$ exists when x→0 and also it is continuous at x =0, *f (x)* is differentiable at x = 0 and has a value 0.

Illustration

Show that the function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , if\ x \neq 0 \\ 0 & , if\ x = 0 \end{cases}$ is differentiable at x=0 and f'(0)=0

Solution

We have,

$$(LHD\ at\ x = 0) = lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0}$$
$$\Rightarrow \lim_{h\to 0} \frac{f(0-h) - f(0)}{0 - h - 0}$$
$$\Rightarrow \lim_{h\to 0} \frac{f(-h) - f(0)}{-h}$$
$$\Rightarrow \lim_{h\to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right) - 0}{-h}$$
$$\Rightarrow \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$
$$\Rightarrow 0 \times (an\ oscillating\ number\ between -1\ and\ 1)$$
$$\Rightarrow 0$$
$$(RHD\ at\ x = 0) = lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0}$$
$$\Rightarrow \lim_{h\to 0} \frac{f(0-h) - f(0)}{0 + h - 0}$$
$$\Rightarrow \lim_{h\to 0} \frac{f(-h) - f(0)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{(h)^2 \sin\left(\frac{1}{-h}\right) - 0}{h}$$

$$\Rightarrow \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow 0 \times (an\ oscillating\ number\ between -1\ and\ 1)$$

$$\Rightarrow 0$$

$$\therefore (LHD\ at\ x=0) = (RHD\ at\ x=0) = 0\,.$$

$$So\ , f(x)\ is\ differentiable\ at\ x=0\ and\ f'(0)=0$$

Illustration

Discuss the differentiability of

$$f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , x \neq 0\ \ at\ x=0 \\ 0 & , x=0 \end{cases}$$

Solution

We have, $f(x) = \begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} = xe^{-2/x} , x \geq 0 \\ xe^{-\left(\frac{-1}{x}+\frac{1}{x}\right)} = x\ , x < 0 \\ 0 \qquad , x = 0 \end{cases}$

Now,

$$(LHD\ at\ x=0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \lim_{x\to 0} \frac{x-0}{x-0} = 1 [\because f(x) = x\ for\ x<0\ and\ f(0)=0\,]$$

And

$$(RHD\ at\ x=0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \lim_{x\to 0} \frac{xe^{-2/x}-0}{x} = 1 \left[\because f(x) = x\ e^{-2/x} for\ x>0\ and\ f(0)=0\,\right]$$

$$\Rightarrow \lim_{x\to 0} e^{-2/x} = 0$$

$$\therefore (LHD\ at\ x=0\,) \neq (RHD\ at\ x=0)$$

$$so\ , f(x)\ is\ not\ differentiable\ at\ x=0$$