Chapter 2 Limit

Day 1

Basic concept

Enquiry:-why do we want to know the limit of a function?

Consider a function $f(x) = \frac{x^2-a^2}{x-a}$. The function f(x) is defined at all points except at x = a because at x = a, f(x) = 0/0 which is indeterminate in the language of mathematics hence we want to know what value does f(x) approach as x approaches a. Graph of $f(x) = \frac{x^2-a^2}{x-a}$ $(x \pm a)$ is shown in fig. 1. Circle at point (a, 2a) means that the point is excluded from the graph of the function. It is for this reason we define limit's get rid of manipulation or simplification constraint of function. So the mineral

We write

$$\lim_{x\to a}\frac{x^2-a^2}{x-a} = \lim_{x\to a}\frac{(x-a)(x+a)}{(x-a)} = \lim_{x\to a}(x+a) = 2a$$

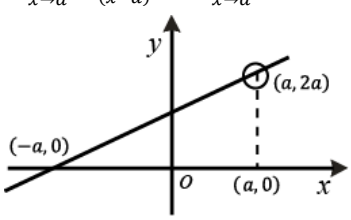
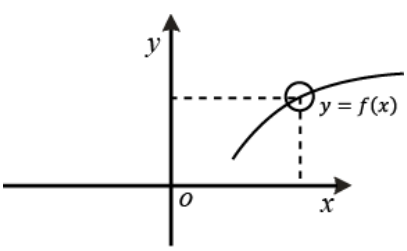


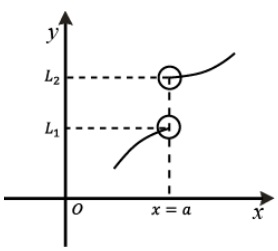
Fig (I)

(i):- Limit of function f(x) is said to exist as $x \to a$ then,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = l \text{ (finite quantity)}$$



(Limit exist, at x = a)



(Limit does not exist, at x = a)

Fundamental algebraic operation on limits of function

Let

$$\lim_{x\to a} f(x) = \alpha \text{ and } \lim_{x\to a} g(x) = \beta \text{ then}$$

(ii):-

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = \alpha \pm \beta$$

$$\lim_{x\to a}[f(x).g(x)] = \lim_{x\to a} f(x) . \lim_{x\to a} g(x) = \alpha\beta$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{\alpha}{\beta} \qquad \text{(proveded } \beta \neq 0)$$

$$\lim_{x\to a} k\, f(x) = k\alpha \text{ where k is a constant.}$$

$$\lim_{x\to a} f[g(x)] = f\left(\lim_{x\to a} g(x)\right) = f(\beta),$$

Provided f is continuous at $g(x) = \beta$.

Standard limits

(a):- $\lim_{x\to 0} \frac{\sin x}{x} = 1$

(b):- $\lim_{x\to 0} \frac{\tan x}{x} = 1 = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x}$. Where x is in radian.

(c):- $\lim_{x\to 0}(1+x)^{1/x} = e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

(d):- $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} \emptyset(x) = \infty$, then

$$\lim_{x\to a}[f(x)]^{\emptyset(x)} = \lim_{x\to a} e^{\emptyset(x)\,[f(x)-1]}$$

(e):- $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a \ (a > 0)$

(f):- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

(g):- $\lim_{x\to a} \frac{x^n - a^n}{x - a} = n\, a^{n-1}$

(h):- $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$

Indeterminate forms

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, 0^0, \infty^0, \infty - \infty \text{ and } 1^\infty, \infty - \infty$$

Sandwich theorem

Let f, g, h be three continuous functions such that f < g < h is an open interval containing ‘a’ and suppose

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$$

Then,

$$\lim_{x \to a} g(x)$$

Also exists and is equal to l.

Some important expansions

(i):- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

(ii):- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \qquad |x| < 1$

(iii):- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + x^n, \qquad n \in N$

(iv):- $a^x = 1 + x\ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \cdots$

(v):- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(vi):- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

(vii):- $(1+x)^{\frac{1}{x}} = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 + \cdots\right]$

(viii):- $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

(ix):- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

(x):- $\sin^{-1} x = x + \frac{x^3}{3!} + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \cdots$

(xi):- $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{6x^6}{6!} + \cdots$

Following strategies should be born in mind for evaluating the limits:

(a):- Factorization

(b):- Rationalization or double rationalization

(c):- Use of trigonometric transformation, appropriate substitution and using standard limits.

Factorization method

Illustration

$$\lim_{x \to 1} \frac{x-1}{2x^2-7x+5}$$

Solution

The given limit

$$\lim_{x \to 1} \frac{x-1}{2x^2-7x+5}$$

$$\Rightarrow \quad \lim_{x \to 1} \frac{x-1}{(x-1)(2x-5)}$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{1}{2x-5}=\frac{1}{2\times 1-5}$$

$$\Rightarrow \quad -\frac{1}{3}$$

Illustration

$$\lim_{x\to 1}\frac{x^3-x^2\ln x+\ln x-1}{x^2-1}$$

Solution

The given limit

$$\lim_{x\to 1}\frac{(x^3-1)-(x^2-1)\ln x}{x^2-1}$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{(x-1)(x^2+x+1)-(x-1)(x+1)\ln x}{(x-1)(x+1)}$$

$$\Rightarrow \quad \lim_{x\to 1}\frac{(x-1)[x^2+x+1-(x+1)\ln x]}{(x-1)(x+1)}$$

$$\Rightarrow \quad \frac{1^2+1+1-(1+1)\ln 1}{(1+1)}$$

$$\Rightarrow \quad \frac{3-0}{2}=\frac{3}{2}$$

<u>Rationalization Method</u>

Illustration

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Solution

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$\Rightarrow \quad \frac{1}{2\sqrt{x}}$$

Illustration

$$\lim_{x\to 2}\frac{\sqrt[3]{(10-x)}-2}{x-2}$$

Solution

Because the fractional power of (10-x) is 1/3, hence we have to multiply and divide the expression by such expression so that each terms become free from radicals (fractional powers). Hence if we think

$\left(\sqrt[3]{(10-x)}\right)-2=a-b$. The suitable terms is (a^2+ab+b^2).

As (a-b) $(a^2+ab+b^2)=a^3-b^3$. Now see below

The given limit

$$\Rightarrow \quad \lim_{x\to 2}\frac{\left(\sqrt[3]{(10-x)}-2\right)\left\{\sqrt[3]{(10-x)}+2.\sqrt[3]{10-x}+4\right\}}{(x-2)\left\{\sqrt[3]{(10-x)}+2.\sqrt[3]{10-x}+4\right\}}$$

$$\Rightarrow \quad \lim_{x\to 2}\frac{\left(\sqrt[3]{(10-x)}\right)^3-2^3}{(x-2)\left\{\sqrt[3]{(10-x)}+2.\sqrt[3]{10-x}+4\right\}}$$

$$\Rightarrow \quad \lim_{x}\frac{(2-x)}{(x-2)\left\{(10-x)^{2/3}+2.(10-x)^{2/3}+4\right\}}$$

$$\Rightarrow \quad -\frac{1}{12}$$

<u>Based on standard formula</u>

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=n\,a^{n-1}, \text{wheere } 'n' \text{is rational number}$$

Illustration

$$lim_{x\to 2}\frac{x^3-2^3}{x-2}$$

Solution

$$lim_{x\to 2}\frac{x^3-2^3}{x-2} \quad \Rightarrow 3(2)^{3-1}$$

$$\Rightarrow 12.$$

Illustration

$$lim_{x\to 1}\frac{\sqrt[3]{x}+\sqrt{x}+x\sqrt{x}-3}{x^3-1}$$

Solution

$$lim_{x\to 1}\frac{\left(\sqrt[3]{x}-1\right)+\left(\sqrt{x}-1\right)+\left(x^{3/2}-1\right)}{(x-1)(x^2+x+1)} \quad \Rightarrow lim_{x\to 1}\left\{\frac{\left(\sqrt[3]{x}-1\right)}{(x-1)}+\frac{\left(\sqrt{x}-1\right)}{(x-1)}+\frac{\left(x^{3/2}-1\right)}{(x-1)}\right\}\left(\frac{1}{(x^2+x+1)}\right)$$

$$\Rightarrow \left\{\frac{1}{3}\times(1)^{-2/3}+\frac{1}{2}\times(1)^{-1/2}+\frac{3}{2}\times(1)^{1/2}\right\}\times\frac{1}{3}$$

$$\Rightarrow 7/9$$

Illustration

$$\lim_{x\to 1}\frac{x^{p+1}-(p+1)x+p}{(x-1)^2}$$

Solution

$$\Rightarrow \lim_{x\to 1}\frac{x^{p+1}-(p+1)x+p}{(x-1)^2} \quad \Rightarrow \quad \lim_{x\to 1}\frac{x^{p+1}-px-x+p}{(x-1)^2} \quad \Rightarrow \quad \lim_{x\to 1}\frac{x(x^p-1)-p(x-1)}{(x-1)^2}$$

Dividing N^r and D^r by (x-1) we get the limit

$$\Rightarrow \quad \lim_{x\to 1}\frac{x\frac{(x^p-1)}{(x-1)}-p}{(x-1)} \quad \Rightarrow \quad \lim_{x\to 1}\frac{x+x^2+x^3+\ldots\ldots+x^p-p}{(x-1)}$$

(Using summation of x terms of G.P.)

$$\Rightarrow \quad \lim_{x\to 1}\frac{(x-1)+\left(x^2-1\right)+\left(x^3-1\right)+\ldots\ldots+\left(x^p-1\right)}{(x-1)}$$