Chapter 1

Fundamentals and Functions

Day 1

Fundamentals

Basic Definitions

(i) Natural numbers:-

N= {1, 2, 3, 4, ………}

(ii) Integers:-

Denoted by I or Z. {……., -3, -2, -1, 0, 1, 2, 3, ……..}

(a) Positive integers by $I^+ = \{1, 2, 3, 4, \ldots \ldots ..\} = natural\ numbers.$

(b) Negative integers by $I^- = \{\ldots \ldots \ldots, -4, -3, -2, -1, \}$

(c) Non-negative integers: {0, 1, 2, 3, 4,……….}=whole numbers

(d) Non-positive integers: {…….., -3, -2, -1, 0}

(iii) Rational Numbers:-

All the numbers of the form p/q, where p and q are integers and $q \neq 0$ are called rational numbers and their set is denoted by Q.

$Q = \frac{p}{q}$ such that $p, q \in I$ and $q \neq 0$ and HCF of p, q is 1.

(1) Every integer is a rational number as it could be written as

$Q = \frac{p}{q}$ (where $q = 1$)

(2) All recurring decimals are rational numbers.

$$Q = \frac{1}{3} = 0.3333 \ldots \ldots$$

(iv)Irrational Numbers:-

Those values which neither terminate nor could be expressed as recurring decimals are irrational numbers. i.e. , it cannot be expressed as p/q form, and are denoted by Q^c (i.e., complement of Q).

$$\sqrt{2},\ 1 + \sqrt{2}, \frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, \frac{2}{\sqrt{2}}, \sqrt{3}, 1 + \sqrt{3}, \pm\frac{1}{\sqrt{3}}, \pi \ldots \ldots \ldots \text{ etc.}$$

(v) Real Numbers:-

The set which contain both rational and irrational are called real and is denoted by R.

$$R = Q \cup Q^c$$

$$\therefore R = \left\{\ldots \ldots . -2, -1, 0, 1, 2, 3, \ldots \ldots .., \frac{5}{6}, \frac{3}{4}, \frac{7}{9}, \frac{1}{3}, \frac{1}{7}, \frac{1}{5}, \ldots \ldots \ldots . \sqrt{2}, \sqrt{3}, \pi, \ldots \ldots .\right\}$$

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CLOSED AND OPEN INTERVALS

(i) Open-open interval:-

$$= a < x < b$$
$$= x \in (a, b)$$
$$= 1 < x < 2 \Rightarrow x \in (1, 2)$$

Illustration

$$-2 < 2x - 1 < 2$$

Solution

$$= -1 < 2x < 3$$
$$= -\frac{1}{2} < x < \frac{3}{2}$$
$$= x \in \left(-\frac{1}{2}, \frac{3}{2}\right)$$

This solution can be graphed on a real line as;

(ii) Open –closed interval:-

$$a < x \leq b \Rightarrow x \in (a, b]$$

(iii) Closed-open interval:-

$$a \leq x < b \Rightarrow x \in [a, b)$$
$$x \in [a, b)$$

(iv) Closed-closed interval:-

$$a \leq x \leq b \Rightarrow x \in [a, b]$$

Illustration

Solve the following equations:

$$\frac{2x-3}{4} + 9 \geq +\frac{4x}{3} + 3$$

Solution

$\Rightarrow$ $\frac{2x-3}{4} - \frac{4x}{3} \geq 3 - 9$

$\Rightarrow$ $\frac{3(2x-3)-16x}{12} \geq -6$

$\Rightarrow$ $\frac{-10x-9}{12} \geq -6$

$\Rightarrow$ $-10x - 9 \geq -72$

$\Rightarrow$ $-10x \geq -63$

"as we know the inequality sign changes, if multiplied by (-ve)"

$\therefore$ $10x \leq 63$

$$= x \leq \frac{63}{10}$$

Hence, the solution set of the given in equation is $(-\infty, 63/10]$.

Modulus or Absolute Value Function

$$y = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Illustration

$|x| = 2$

Solution

$|x| = 2 \Rightarrow x = \pm 2,$

Which means, x is at a distance of 2 units from 0 which is certainly 2 and -2.

Illustration

$|x| = -2$

Solution

$|x| = -2 \Rightarrow x\ has\ no\ solution.$

As |x| is always positive or zero it can never be negative.

$\therefore$ $\text{R.H.S} > \text{L.H.S}$

Illustration

$|x| < 2$

Solution

$|x| < 2$

It means that x is the number, which is at distance less than 2 from 0.

Illustration

$|x| < -2$

Solution

$|x| < -2$

Which shows no solution.

As L.H.S. is non-negative and RHS is negative or |x|<-2 does not possess any solution.

Illustration

$|x| > -2$

Solution

$|x| > -2$

We knows here L.H.S. ≥ 0 and R.H.S < 0

$\Rightarrow$ $\text{L.H.S.} > \text{R.H.S.}$

i.e., above statement is true for all real x.

(as we know that non-negative number is always greater than negative.)

Illustration

$|x| > 2$

Solution $|x| > 2$

It means that x is the number which is at distance greatest than 2 from 0.

$x < -2$ or $x > 2.$

<u>Generalised Results</u>

(i):- For any real number x, we have

$$x^2 = |x|^2$$

(ii):- For any real number x, we have

$$\sqrt{x^2} = |x|$$

(iii):- If a > 0 then:

(a) $x^2 \le a^2 \Leftrightarrow |x| \le a \Leftrightarrow -a \le x \le a$

(b) $x^2 < a^2 \Leftrightarrow |x| < a \Leftrightarrow -a < x < a$

(c) $x^2 \ge a^2 \Leftrightarrow |x| \ge a \Leftrightarrow x \le -a \; or \; x \ge a$

(d) $x^2 > a^2 \Leftrightarrow |x| > a \Leftrightarrow x < -a \; or \; x > a$

(e) $a^2 \le x^2 \le b^2 \Leftrightarrow a \le |x| \le b \Leftrightarrow x \in [-b, -a] \cup [a, b]$

(f) $a^2 < x^2 < b^2 \Leftrightarrow a < |x| < b \Leftrightarrow x \in (-b, -a) \cup (a. b)$

(iv):- If a < 0 then:

(a) $|x| \le a \Rightarrow$ no solution.

(b) $|x| \ge a \Rightarrow$ all real numbers.

(v):- |x + y| = |x| + |y|

$\Leftrightarrow (x \ge 0 \text{ and } y \ge 0) \text{ or } (x \le 0 \text{ and } y \le 0)$

$\Leftrightarrow xy \ge 0.$

(vi):- $|x - y| = |x| - |y|$

$\Leftrightarrow (x \ge 0, y \ge 0 \text{ and } |x| \ge |y|) \text{ or } (x \le 0, y \le 0 \text{ and } |x| \ge |y|).$

(vii):- $|x \pm y| \le |x| + |y|$

(viii):- $|x \pm y| \ge |x| - |y|$

Illustration

solve $|x - 1| \le 2$

Solution

$\Rightarrow \quad |x - 1| \le 2$

$\Rightarrow \quad -2 \le x - 1 \le 2$

$\Rightarrow \quad -2 + 1 \le x \le 2 + 1$

$\Rightarrow \quad -1 \le x \le 3$

$x \in [-1, 3].$

Illustration

$1 \le |x - 1| \le 3$

Solution

$\Rightarrow \quad -3 \le (x - 1) \le -1 \text{ or } 1 \le (x - 1) \le 3$

i.e., the distance covered is between 1unit to 3 units.

$\Rightarrow \quad -2 \leq x \leq 0$ or $2 \leq x \leq 4$

Hence, the solution set of the given in equation is $x \in [-2, 0] \cup [2, 4]$

Illustration

$|x-1| \leq 5, |x| \geq 2$

Solution

$i.e., (-5 \leq x - 1 \leq 5)$ and $(x \leq -2$ or $x \geq 2)$

$i.e., (-4 \leq x \leq 6)$ .....(i) and

$(x \leq -2$ or $x \geq 2)$ ... (ii)

(i) and (ii) could be graphically shown as;

Thus, the shaded portion i.e., common to both (i) and (ii) is the required region.

$\Rightarrow \quad x \in [-4, 2] \cup [2, 6].$

Illustration

$|x-1| + |x-2| \geq 4$

Solution

On the L.H.S. of the given in equation we have two modulus,

So we should define each

Modulus, i.e., by equating to zero.

$$|x-1| = \begin{cases} (x-1), & x \geq 1 \\ -(x-1), & x < 1 \end{cases}$$

$$|x-2| = \begin{cases} (x-2), & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$$

Thus it gives three cases:

Case I:- When $-\infty < x < 1$

$|x-1| + |x-2| \geq 4$

$\Rightarrow \quad -(x-1) - (x-2) \geq 4$

$\Rightarrow \quad -2x + 3 \geq 4$

$\Rightarrow \quad -2x \geq 1$

$\Rightarrow \quad x \leq -\frac{1}{2}$ ... (i)

But $-\infty < x < 1$

$\therefore$ solution set is $x \in \left(-\infty, \frac{1}{2}\right]$

Case II:-

When $1 \leq x \leq 2$

$|x-1| + |x-2| \geq 4$

$\Rightarrow \quad (x-1) - (x-2) \geq 4$

$\Rightarrow \quad 1 \geq 4,$ which is meaningless.

$\therefore \quad$ no solution for $x \in [1, 2]$ ... (ii)

Case III:-

When x >2

$|x-1|+|x-2| \geq 4$

$\Rightarrow \quad (x-1)+(x-2) \geq 4$

$\Rightarrow \quad 2x-3 \geq 4$

$\Rightarrow \quad x \geq \frac{7}{2}$

But $x > 2$

$\therefore$ Solution set is $\left[\frac{7}{2}, \infty\right)$ ... (iii)

From (i), (ii) and (iii), we get,

$x \in \left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{7}{2}, \infty\right)$.

Wavy Curve Method/Number Line Rule/ Sign Scheme For Rational Function

It is used to solve algebraic inequalities using following steps:

(i):- Put only odd power factors in numerator and denominator equal to zero separately. (as for polynomial function only numerator =0, denominator $\neq 0$).

(ii):- Plot these points on number line in increasing order.

(iii):- Now check the coefficients of x and make them positive.

(iv):- Start the number line form right to lift taking sign of $f(x)$.

(v):- Check your answer should not contain a point for which $f(x)$ does not exists.

Illustration

Find the interval in which $f(x)$ is positive or negative,

$f(x) = (x-1)(x-2)(x-3)$

Solution

Here, $f(x) = (x-1)(x-2)(x-3)$ has all factors with odd powers so put them zero.

i.e., $x-1=0, x-2=0, x-3=0$,

We get,

x = 1, 2, 3

Plotting on number line we get,

$f(x) > 0$ when $1 < x < 2$ and $x > 3$ $\quad f(x) < 0$ when $x < 1$ and $2 < x < 3$

Illustration

Find the values of x for which

$f(x) = \frac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)} \leq 0$

Solution

Here,

$f(x) = \frac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)} \leq 0 \quad f(x) = -\frac{(x-2)^2(x-1)(x-3)^3(x-4)^2}{(x+1)} \geq (x \neq -1)$

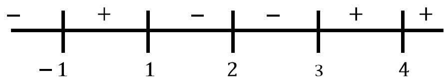
Putting zero to $(x-1), (x-3)^3, (x+1)$ as having odd powers and neglecting $(x-2)^2, (x-4)^2$.

We get,

$f(x) \leq 0$ when $-1 < x \leq 1$ $\{$using (i) as $x \neq -1\}$

$= 3 \leq x < \infty$

$= x \in (-1, 1] \cup [3, \infty)$



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Fundamentals Of Quadratic Equations

We know every quadratic equation represents a parabola (explained in later part).

Thus,

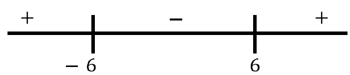
(i):- If $ax^2 + bx + c > 0, \forall\, x \in R \Rightarrow a > 0$ and $D < 0$

(ii):- If $ax^2 + bx + c < 0, \forall\, x \in R \Rightarrow a < 0$ and $D < 0$

Illustration

Find a for which

$3x^2 + ax + 3 > 0, \forall\, x \in R$?



Solution

Here,

$3x^2 + ax + 3 > 0, \forall\, x \in R$

$\Rightarrow \quad D < 0$

$\Rightarrow \quad (a)^2 - 4(3)(3) < 0$

$\Rightarrow \quad a^2 - 36 < 0$

$\Rightarrow \quad (a-6)(a+6) < 0,$

Using number line rule as shown in figure,

Which shows

$-6 < a < 6$

$a \in (-6, 6)$

Illustration

$x^2 = 4$

Solution

$x^2 = 4$

$\Rightarrow \quad (x+2)(x-2) = 0$

$\Rightarrow \quad x = 2, -2.$

Illustration

$x^2 = -4$

Solution

$x^2 = -4$

Hence, no solution as $\text{L.H.S.} \geq 0$and R.H.S.< 0.

Illustration

$x^2 < 4$

Solution

$x^2 < 4$

$\Rightarrow \quad (x-2)(x+2) < 0,$

Using number line rule as shown in figure, we get , $-2 < x < 2$.

Illustration

$x^2 > 4$

Solution

$x^2 > 4$

$\Rightarrow \quad (x-2)(x+2) > 0,$

Using number line rule as shown in figure we get; -2 or $x > 2$.

Illustration

$x^2 < -4$

Solution

$x^2 < -4$

$\Rightarrow \quad x^2 + 4 < 0, \textit{which is not possible}$

$\therefore$ no solution .

Illustration

$x^2 > -4$

Solution

$x^2 > -4$

$\Rightarrow \quad x$

$\Rightarrow \quad x^2 + 4 > 0,$ which is true for all

$x \in$ real number

$\therefore \quad x \in R.$

<u>The sum of several non-negative terms</u>

Illustration

Solve:-

$(x+1)^2 + (x^2 + 3x + 2)^2 = 0$

Solution

Here,

$(x+1)^2 + (x^2 + 3x + 2)^2 = 0$

If and only if each term is zero simultaneously,

$(x+1) = 0$ and $(x^2 + 3x + 2) = 0$

$x = -1$ and $x = -1, -2$

$\therefore$ The common solution is $x = -1$

Hence, solution of above equation is x = -1.

Illustration

Solve:-

$$|x+1| + \sqrt{x-1} = 0$$

Solution

Here

$|x+1| + \sqrt{x-1} = 0,$

Where each term is non-negative

$\therefore$ $|x+1| = 0$ and $\sqrt{x-1} = 0$

Should be zero simultaneously

i.e., x = -1 and x =1, which is not possible.

There is no x for which each term is zero simultaneously.

Hence, there is no solution.

Illustration

Solve:-

$$|x^2 - 1| + (x-1)^2 + \sqrt{x^2 - 3x + 2} = 0$$

Solution

Here each of the terms is non-negative thus each must be zero simultaneously,

$(x^2 - 1) = 0, (x-1)^2 = 0$ and $x^2 - 3x + 2 = 0$

$\Rightarrow$ $x = \pm 1, x = 1$ and $x = 1, 2$

The common solution is x=1

Therefore, x=1 is solution of above equation.

Illustration

Let

$f(x) = x$ and $g(x) = |x|$ be two real valued functions,

$\emptyset(x)$be a function satisfying the condition ;

$[\emptyset(x) - f(x)]^2 + [\emptyset(x) - g(x)]^2 = 0.$ Then find $\emptyset(x)$

Solution

Here,

$[\emptyset(x) - f(x)]^2 + [\emptyset(x) - g(x)]^2 = 0$ is only possible if

$\emptyset(x) - f(x) = 0$ and $\emptyset(x) - g(x) = 0$

$\Rightarrow$ $\emptyset(x) = f(x) = g(x)$

$\emptyset(x) = x = |x|,$

Which is only possible if x is non-negative

Therefore,

$\emptyset(x) = x, \forall\, x \in [0, \infty)$

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