## Ellipse

## Day - 1

## 1. Ellipse Definition

An ellipse is the locus of a point which moves in a plane such that its distance from a foxed point (i.e., focus) is a constant ratio from a fixed line (i.e., directrix). This ratio is called eccentricity and is denoted by $e$. For an ellipse $e<1$.


### 1.1 Standard Equation of Ellipse

The focus $S$ is (CS, 0) i.e., (ae, 0 )
Now draw PM $\perp$ MZ

$$
\begin{aligned}
& \frac{S P}{P M}=e \\
& \text { or } \\
& (S P)^{2}=e^{2}(P M)^{2} \\
& (x-a e)^{2}+(y-0)^{2}=e^{2}\left(\frac{a}{e}-x\right)^{2} \\
& \Rightarrow \quad(x-a e)^{2}+y^{2}=(a-e x)^{2} \\
& \Rightarrow \quad x^{2}+a^{2} e^{2}-2 a e x+y^{2}=a^{2}-2 a e x+e^{2} x^{2} \\
& \Rightarrow \quad x^{2}\left(1-e^{2}\right)+y^{2}=a^{2}-2 a e x+e^{2} x^{2} \\
& \Rightarrow \quad \frac{x^{2}}{a^{2}}+\frac{y^{2}}{a^{2}\left(1-e^{2}\right)}=1 \\
& \text { or } \\
& \frac{\mathrm{x}^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \quad \text { where } b^{2}=a^{2}\left(1-e^{2}\right)
\end{aligned}
$$

This is the standard equation of an ellipse, $\mathrm{AA}^{\prime}$ and $\mathrm{BB}^{\prime}$ are called the major and minor axes of the ellipse. (Here $b<a$ ) and A and A' are the vertices of the ellipse.

### 1.2 Generally

The equation to ellipse, whose focus is the point $(h, k)$ and directrix is $l x+m y+n=0$ and whose eccentricity is e, isds

$$
(x-h)^{2}+(y-k)^{2}=e^{2}, \frac{(l x+m y+n)}{l^{2}+m^{2}}
$$

### 1.3 Some Terms Related to an Ellipse

Let the equation of the ellipse

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1,(a>b)
$$



## (1) Centre

All chords passing through C is bisected at C
Here $\mathrm{C} \equiv(0,0)$
(2) Foci

S and S' are two foci of the ellipse and their co -ordinations are (ae, 0) and ( $-\mathrm{ae}, 0$ ) respectively.

## (3) Directrices

ZM and Z' M' are two Directrices of the ellipse and their equation are $e x=\frac{a}{e}$ and $x=-\frac{a}{e}$ respectively.

## (4) Axes

The lines AA' and BB' are called the major and minor axes of the ellipse

| $\therefore$ | $0<\mathrm{e}<1$ |  |
| :--- | :--- | :--- |
| or | $0<\mathrm{e}^{2}<1$ | $\left(\therefore 0>-\mathrm{e}^{2}>-1\right)$ |
| or | $0<1-\mathrm{e}^{2}<1$ | $\left(\right.$ or $\left.1>1-\mathrm{e}^{2}>1-1\right)$ |
| or | $\mathrm{a}^{2}\left(1-\mathrm{e}^{2}\right) \mathrm{a}^{2}$ | $\left(\right.$ or $\left.0<1-\mathrm{e}^{2}<1\right)$ |
| or | $\mathrm{b}^{2}<\mathrm{a}^{2}$ |  |
| or | $\mathrm{b}<\mathrm{a}$ |  |

## (5) Double Ordinates

If P be a point on the ellipse draw PN perpendicular to the axis of the ellipse and produced to meet the curve again $\mathrm{P}^{\prime}$. Then $\mathrm{PP}{ }^{\prime}$ is called a double ordinate. If abscissa of P is h then ordinate of P ,

$$
\begin{array}{lll}
\therefore & \frac{y^{2}}{b^{2}}=1-\frac{h^{2}}{a^{2}} & \\
\text { or } & y=\frac{b}{a} \sqrt{\left(a^{2}-h^{2}\right)} & \text { (for first quadrant) } \\
\text { or and ordinate of } \mathrm{P}^{\prime} & y=-\frac{b}{a} \sqrt{\left(a^{2}-h^{2}\right)} & \text { (for fourt quadrant) }
\end{array}
$$

or Hence, co - ordination of P and $\mathrm{P}^{\prime}$ are

$$
\left(h, \frac{b}{a} \sqrt{a^{2}-h^{2}}\right) \text { and }\left(h,-\frac{b}{a} \sqrt{a^{2}-h^{2}}\right) \text { respectively }
$$

## (6) Latus rectum

The double ordinates $L L^{\prime}$ and $L_{1} L_{1}{ }^{\prime}$ are latus - rectums of the ellipse. These line are perpendicular to major axis A' A and through thr foci $S$ and $S^{\prime}$ respectively.

### 1.4 Length of the Latus Rectum

Now let

$$
\mathrm{LL}^{\prime}=2 \mathrm{k}
$$

Then

$$
\mathrm{LS}=\mathrm{L}^{\prime} \mathrm{S}=\mathrm{k}
$$

Co -ordinations of L and $\mathrm{L}^{\prime}$ are $(\mathrm{ae}, \mathrm{k})$ and $(\mathrm{ae},-\mathrm{k})$ lies on the ellipse

$$
\begin{aligned}
& \frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \\
& \therefore \quad \frac{a^{2} e^{2}}{a^{2}}+\frac{k^{2}}{b^{2}}=1 \\
& \text { or } \quad k^{2}=b^{2}\left(1-e^{2}\right) \\
& =b^{2}\left(\frac{b^{2}}{a^{2}}\right) \quad\left\{\therefore b^{2}=a^{2}\left(1-e^{2}\right)\right\} \\
& \therefore \quad k=\frac{b^{2}}{a} \quad(\because k>0) \\
& \therefore \quad 2 k=\frac{2 b^{2}}{a}=L L^{\prime}
\end{aligned}
$$

$\therefore$ Length of latus rectum $L L^{\prime}=L_{1} L_{1}^{\prime}=\frac{2 b^{2}}{a}$ and end of point of latus - rectum $a$

$$
\begin{array}{ll}
L \equiv\left(a e, \frac{b^{2}}{a}\right) ; & L^{\prime} \equiv\left(a e,-\frac{b^{2}}{a}\right) \\
L_{1} \equiv\left(-a e, \frac{b^{2}}{a}\right) ; & L_{1}^{\prime} \equiv\left(-a e,-\frac{b^{2}}{a}\right)
\end{array}
$$

## (7) Focal chord

A chord of the ellipse passing through its focus is called a focal chord.

## (8) Vertices

The vertices of the ellipse are the points where the ellipse meets its major axis.
Hence A and A' are the vertices
$\therefore \quad \mathrm{A} \equiv(a, 0)$ and $\mathrm{A}^{\prime} \equiv(-a, 0)$

## (9) Parametric equation of the ellipse

The circle described on the major axis of an ellipse as diameter is called the auxiliary circle of the ellipse. Let the equation of ellipse be

$$
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1
$$


$\therefore$ Equation of its auxiliary circle is $x_{2}+y_{2}=a_{2}\left(\because \mathrm{AA}^{\prime}\right.$ is diameter of the circle $)$
Let Q be a point on the auxiliary circle $x_{2}+y_{2}=a_{2}$ such that QP produced is perpendicular to the x - axis.

Then P and Q are the corresponding points on the ellipse and the auxiliary circle respectively.
Let $\angle \mathrm{QCA}=\phi$
$(0 \leq \phi<2 \pi)$
Ie., the eccentric angle of P on an ellipse is the angle which the radius (or radius vector) through the corresponding point on the auxiliary circle makes with the major axis.

$$
\therefore \quad \mathrm{Q} \equiv(\mathrm{a} \cos \phi, \mathrm{a} \sin \phi)
$$

$\therefore$ Now $\mathrm{x}-\mathrm{co}-$ ordination of P is a $\cos \phi$
Let y - co-ordination of P is y then $(\mathrm{a} \cos \phi, \mathrm{y})$ lies on the ellipse

$$
\begin{array}{ll} 
& \frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \\
& \frac{a^{2} \cos ^{2} \phi}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \\
\Rightarrow & y^{2}=b^{2} \sin ^{2} \phi \\
\therefore & y= \pm b \sin \phi
\end{array}
$$

$\therefore \mathrm{P}$ is in I quadrant
$\therefore \quad y=b \sin \phi$
Co - ordinates of P are $(a \cos \phi, b \sin \phi)$. We hane $x=a \cos \phi, y=b \sin \phi$ are called parametric equations of the ellipse.
This point $(a \cos \phi, b \sin \phi)$ is also called the point' $\phi$ '.

### 1.5 Focal Distance of a Point

The sum of focal distance of any point on the ellipse is to the major axis.
The ellipse is

$$
\begin{equation*}
\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1 \tag{1}
\end{equation*}
$$

The foci $S$ and $S^{\prime}$ are $(a e, 0)$ and $(-a e, 0)$.
The equations of its Directrices MZ and M'Z' are $x=a / e$ and $x=-a / e$
Let $\mathrm{P}\left(x_{1}, y_{1}\right)$ be any point on (1)
Now

$$
\begin{aligned}
\mathrm{SP} & =e \mathrm{PM}=e\left(\frac{a}{e}-x_{1}\right) \\
& =a-\mathrm{e} x_{1} \\
\mathrm{~S}^{\prime} \mathrm{P} & =e P M^{\prime}=e\left(\frac{a}{e}+x_{1}\right) \\
& =a+\mathrm{e} x_{1} \\
\mathrm{SP}+\mathrm{S}^{\prime} \mathrm{P} & =\left(a-\mathrm{e} x_{1}\right)+\left(a+\mathrm{e} x_{1}\right) \\
& =2 a=\mathrm{AA}^{\prime}=\text { major axis }
\end{aligned}
$$

and


Hence the sum of the focal distance of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse

### 1.6 Another Definition of Ellipse

An ellipse is the locus of a point which moves in a plane such that the sum of its distance from two fixed points in the same plane is always constant.
Note :- SP and SP' are also called focal radii of the ellipse

$$
\mathrm{SP}=a-\mathrm{e} x_{1} \text { and } \mathrm{S}^{\prime} \mathrm{P}=a+\mathrm{e} x_{1} .
$$

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