Chapter 5 Parabola

Day – 1

1. Conic Section

Definition

The locus of a point which moves in a plane such that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is always constant is known as conic section.

The fixed point is called a focus of the conic and this fixed line is called the directrix of the conic, also this constant ratio is called the eccentricity of the conic and is denoted by e.

If

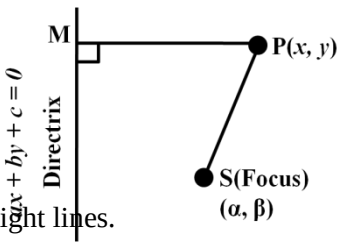
$e = 1, \Rightarrow$ Parabola

$e < 1 \Rightarrow$ Ellipse

$e > 1 \Rightarrow$ Hyperbola

$e = 0 \Rightarrow$ Circle

$e = \infty \Rightarrow$ Pair of straight lines.



In the figure:

$$\frac{SP}{PM} = cons = e$$

$$SP = e PM$$

1.1 Equation of Conic Section

If the focus is (α, β) and the directrix is ax + by + c = 0 then the equation of the conic section whose eccentricity is e

$$PS = e PM$$

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = e\frac{|ax+by+c|}{\sqrt{a^2+b^2}}$$

or $$(x-\alpha)^2 + (y-\beta)^2 = e^2\frac{(ax+by+cx)^2}{a^2+b^2}$$

1.2 Some Important Terms

1. Axis

The straight line passing through the focus and parallel to the directrix is called the axis of conic section.

2. Vertex

The points of intersection of the conic section and the axis is (are) called vertex (vertices) of conic section.

3. Focal Chord

Any chord passing through the focus is called focal chord.

4. Double Ordinate

A straight line drawn perpendicular to the axis and terminated at both end of the curve is a double ordinate.

5. Latus Rectum (LR)

The double ordinate passing through the focus is called latus rectum.

6. Centre

The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

Illustration

Find the locus of a point, which moves such that its distances from the point (0, –1) is twice its distance from the line 3x + 4y + 1 = 0.

Solution

Let P(x_1, y_1) be the point is required.

Its distance from (0, –1) = 2 × its distance from the line 3x + 4y + 1 = 0

$$\Rightarrow \qquad \sqrt{(x_1-0)^2+(y_1+1)^2} = 2 \times \frac{|3x_1+4y_1+1|}{\sqrt{(3^2+4^2)}}$$

$$5\sqrt{x_1^2+(y_1+1)^2} = 2|3x_1+4y_1+1|$$

Squaring and simplifying, we get

$$25(x_1^2+y_1^2+2y_1+1) = 4(9x_1^2+16y_1^2+24x_1y_1+6x_1+8y_1+1)$$

or $$11x_1^2+39y_1^2+96x_1y_1+24x_1-18y_1-21=0$$

Hence the locus (x_1, y_1) is

$$11x_1^2+39y_1^2+96x_1y_1+24x_1-18y_1-21=0$$

1.3 Recognisation of Conics

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

Case I $\Delta = 0$ Degenerate Conics

Case II $\Delta \neq 0$ Non-Degenerate Conics

Conditions	Nature of Conics
$\Delta \neq 0$, $h = 0$, $a = b$	Circle
$\Delta \neq 0$ $ab - h_2 = 0$	A Parabola
$\Delta \neq 0$ $ab - h_2 > 0$	A Ellipse or empty set
$\Delta \neq 0$ $ab - h_2 < 0$	A Hyperbola
$\Delta \neq 0$ $ab - h_2 < 0$ $a + b = 0$	A Rectangular Hyperbola

Illustration

What conic does $13x^2 - 18y + y^2 + 2x + 14y - 2 = 0$ represents.

Solution

Compare the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$\therefore \quad a = 13,\ h = -9,\ b = 37,\ g = 1,\ f = 7,\ c = -2$

Then $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= (13)(37)(-2) + 2(7)(1)(-9) - 13(7)^2 - 37(1)^2 + 2(-9)^2$

$= -962 - 126 - 637 - 37 + 162$

$= 1600 \neq 0$

and also $h^2 = (-9^2 = 81)$ and $ab = 13 \times 37 = 481$

here $ab - h^2 = 400 > 0$

So we have $ab - h^2 > 0$ and $\Delta \neq 0$.

Illustration

$x^2 - y^2 - 2x + 2y + \lambda = 0$ represent a degenerate conic. Find λ.

Solution

For degenerate conic $\Delta = 0$ Compare the given equation with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore \quad a = 1,\ b = -1,\ h = 0,\ g = -1,\ f = 1,\ c = \lambda$

Then $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$(1)(-1)(\lambda) + 0 - 1 \times (1)^2 + 1(-1^2) - \lambda(0)^2 = 0$

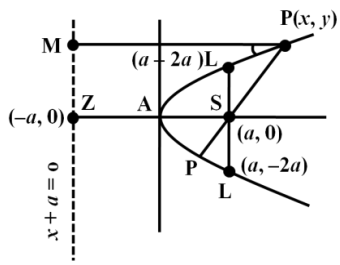
$-\lambda - 1 + 1 = 0 \quad \Rightarrow \quad \lambda = 0$

1.4 Standard Equation of Parabola

Let S be the focus and Z be the directrix of the parabola. Draw SZ perpendicular to ZM, let A be the mid - point of XZ, then as

AS = AZ

So A lies on the parabola. Takes A as the origin and a line AY through A perpendicular to AX as Y-axis.



Let AS = AZ = a (> 0)

Then co-ordinate of S is (a, 0) and the equation of ZM is

$x = -a \quad or \quad x + a = 0$

Then $SP = \sqrt{(x-a)^2 + (y-0)^2}$

$SP = \sqrt{(x-a)^2 + y^2}$

Not for the parabola

SP = PM

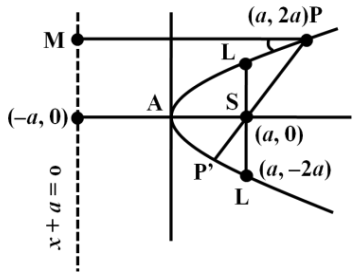
or $(SP)^2 = (PM)^2$

$(x-a)^2 + y^2 = (a+x)^2$

$y^2 = (a+x)^2 - (x-a)^2 = 4ax$

$\therefore \quad y^2 = 4ax$

Which is the required equation of the parabola.



1.5 Some Terms Related to the Parabola

1. Equation $y^2 = 4ax$

or $y^2 = (\text{Lotus Rectum}) \times x$

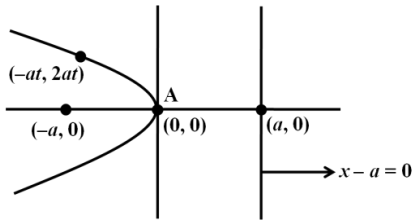
2. A is the midpoint of Z and S so, AZ = AS.
3. Directrix is always perpendicular to the axis of the parabola.
4. Parabola is always symmetric about its axis.
5. Latus Rectum L.R. = 4*a*. (end point (*a*, 2*a*), (*a*, –2*a*)).
6. Focal chord: A chord of parabola which passes through focus is a ***Focal chord.***
In objective questions use LL' as Focal chord in subjective questions use PP' as Focal chord.
7. Focal Distance: The Focal distance from the focus i.e., SP.

$$SP = PM = |x + a|$$

8. Parametric Equation

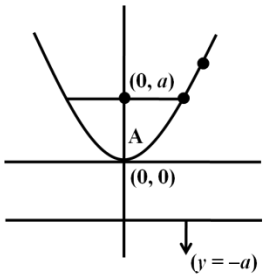
$$x = at_2,\ y = 2at.$$

Other forms of parabola with latus rectum 4*a*



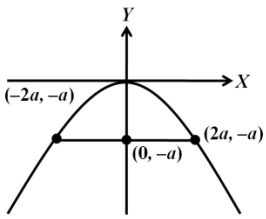
1. Parabola opening to left $(y^2 = 4ax)a > 0$
(i) Vertex is A(0, 0)
(ii) Focus is S (–*a*, 0)
(iii) Equation of directrix is x – a = 0
(iv) Equation of the axis is y = 0
(v) Equation of tangent at vertex is x = 0
(vi) Length of latus rectum is 4a.
(vii) Extremities of latus rectum are L(–*a*, 2*a*) and L'(–*a*, –2*a*)
(viii) Equation of latus rectum is $x = -a$ i.e., $x + a = 0$
(ix) Parametric co-ordinates is $(-at^2, 2at)$

2. Parabola opening upward. $x^2 = 4ay$, $a > 0$



(i) Vertex is A(0, 0)
(ii) Directrix $y + a = 0$
(iii) Focas (0, *a*)
(iv) L.R. = 4*a*
(v) Focal distance = $|y + a|$.
(vi) End point of L.R. = (2*a*, *a*)
(vii) Axis of the parabola $x = 0$
(viii) Parametric co-ordinate $(2at, at^2)$

3. Parabola opening downwards $x^2 = -4ay$



(i) Vertex is A(0, 0)
(ii) S (0, –*a*)
(iii) Directrix $y - a = 0$
(iv) L.R. = 4*a*
(v) Focal distance = $|y - a|$.
(vi) End point of L.R. = (2*a*, –*a*), (–2*a*, –*a*) (vii) Parametric $(2at, -at^2)$

Question Practice Online