Chapter 4

Circle

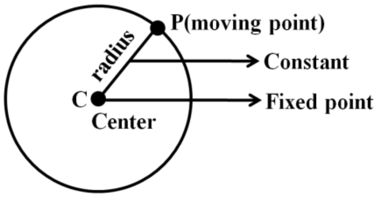
Day – 1

1. Circle

Definition

A circle is a locus of a point which moves in a plane, so that its distance from a fixed point in the plane is always constant.

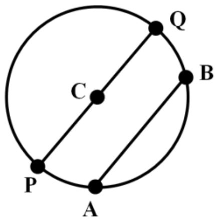
The fixed point is called the centre of the circle and the constant distance is called its radius.



1.1 Equation of a Circle

The curve traced by the moving point is called its circumference.

i.e., the equation of any circle is satisfied by coordinates of all points on its circumference.



1.2 Chord and Diameter

Chord

The line joining any two points on the circle is called a chord.

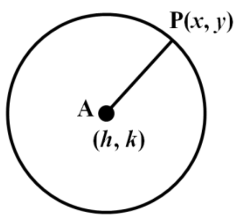
Diameter

A chord passing through is called diameter.

AB = Chord, PQ = Diameter

1.3 Equation of Circles in Different Forms

1. Centre – Radius Form



Let 'a' be radius and C (h, k) be the centre of any circle.

If P (x, y) be any point of the circle.

$$(x-h)^2+(y-k)^2=a^2$$

Note: C(0, 0)

$$x^2+y^2=0$$

2. Parametric Form

Note: $(x-h)^2+(y-k)^2=a^2$

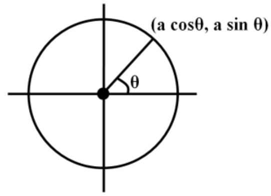
$x-h=a\cos\theta,\ \ y-k=a\sin\theta$

$\Rightarrow \quad x=h+a\cos\theta,\ \ y=k+a\sin\theta$

$x^2+y^2=a^2$

Parametric $\Rightarrow\ x=a\cos\theta,\ y=a\sin\theta$

$0\le\theta<2\pi$



Note:- Equation of the chord joining $(a\cos\alpha,\ a\sin\alpha)\ and\ (a\cos\beta,\ a\sin\beta)$ of $x^2+y^2=a^2$

$$x\cos\left(\frac{\alpha+\beta}{2}\right)+y\sin\left(\frac{\alpha+\beta}{2}\right)=a\cos\left(\frac{\alpha-\beta}{2}\right)$$

1.4 General Form

$(x-h)^2+(y-k)^2=a^2 \qquad C(h,k)$

$x^2+y^2-2hx-2ky+h^2+k^2-a^2=0$

$x^2+y^2+2gx+2fy+c=0$

$h=-g,\ \ k=-f,\ \ a=\sqrt{g^2+f^2-c}$

Coordinates of the centre

$C(-g,-f)$ Radius $=\sqrt{g^2+f^2-c}$

1.5 Note 1. Rule For Finding the Centre and Radius of a Circle

(i) Make the coefficient of x^2 and y^2 equal to 1 and right hand R.H.S. = 0.

(ii) Then coordinate of centre $\left(-\frac{1}{2}\text{ cofficient of } x, -\frac{1}{2}\text{ cofficient of } y\right)$

(**iii**) Radius $=\sqrt{g^2+f^2-c}$

2. Conditions For a Circle

A general equation of second degree $ax^2+2hxy+by^2+2gx+2fy+c=0$ in x, y represent a circle if,

(i) $\Delta\neq 0$

(ii) Coefficient of x^2 = coefficient of y^2

(iii) Coefficient of $x\,y$ = 0.

3. Nature of the Circle

Radius of a circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{g^2+f^2-c}$

Now following cases are possible.

(i) If $g^2+f^2-c>0$ than a (radius of the circle) is real, hence a real circle is possible.

(ii) If $g^2+f^2-c=0$ than a (radius of the circle) is real. Hence in this case it is called a point circle.

(iii) If $g^2+f^2-c<0$ radius is imaginary ⇒ Virtual or imaginary circle.

4. Concentric Circle

Two circles having the same centre C (h, k) but different radii r_1 and r_2 respectively are called concentric circle. Then for the equation of concentric circles differ only in constant term.

Illustration

Find the centre and radius of the circle.

$$2x^2 + 2y^2 = 3x - 5y + 7$$

Solution

Given equation of circle is

$$2x^2 + 2y^2 = 3x - 5y + 7$$
$$x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - \frac{7}{2} = 0$$

If centre is (α, β), then $\alpha = -\frac{1}{2}\left(-\frac{3}{2}\right) = \frac{3}{4}$

and $\beta = -\frac{1}{2}\left(\frac{5}{2}\right) = -\frac{5}{4}$

$\therefore$ centre of circle is (α, β) $i.e.,$ $\left(\frac{3}{4}, -\frac{5}{4}\right)$

and radius of the circle $= \sqrt{\alpha^2 + \beta^2 - (\text{constant term})}$

$$= \sqrt{\frac{9}{16} + \frac{25}{16} + \frac{7}{2}} = \sqrt{\frac{9+25+56}{16}} = \frac{3\sqrt{10}}{4}$$

Illustration

Prove that the radii of the circles are A.P. $x^2 + y^2 = 1,\ x^2 + y^2 - 2x - 6y = 6,$ $x^2 + y^2 - 4x - 12y = 9.$

Solution

Given circle are

$x^2 + y^2 = 1$ ... (i)

$\Rightarrow$ $x^2 + y^2 - 2x - 6y - 6 = 0$ ... (ii)

and $x^2 + y^2 - 4x - 12y - 9 = 0$ ... (iii)

Let r_1, r_2, r_3 be the radii of the circles (1), (2) and (3) respectively.

then $r_1 = 1$ $r_2\sqrt{(-1)^2 + (-6)^2 + 9} = 4$

and $r_3 = \sqrt{(-2)^2 + (-6)^2 + 9} = 7$

Clearly, $r_2 - r_1 = 4 - 1 = 3 = r_3 - r_2$

Hence r_1, r_2 and r_3 are in A.P.

Illustration

Find the equation of circle whose centre is the point of intersection of the lines. $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ and passes through origin.

Solution

The point of intersection of the lines $2x – 3y + 4 = 0$ and $3x + 4y – 5 = 0$ is $\left(-\frac{1}{17}, \frac{22}{17}\right)$

Therefore, the centre of the circle is at $\left(-\frac{1}{17}, \frac{22}{17}\right)$

Since the origin lies on the circle, its distance from the centre of the circle is radius of the circle, therefore

$$r = \sqrt{\left(-\frac{1}{17} - 0^2\right) + \left(\frac{22}{17} - 0\right)^2} = \sqrt{\frac{485}{289}}$$

$\therefore$ The equation of the circle becomes

$$\left(x + \frac{1}{17}\right)^2 + \left(y - \frac{22}{17}\right)^2 = \frac{485}{289}$$

$$17(x^2+y^2)+2x-44y=0$$

Illustration

Find the equation of the circle concentric with the circle $x^2+y^2-8x+6y-5=0$ and passing through (–2, –7).

Solution

The given equation of circle is

$$x^2+y^2-8x+6y-5=0$$

Therefore, the centre of the circle is at (4, – 3). Since the required circle is concentric with this circle, therefore, the centre of the required circle is also at (4, – 3). Since the point (–2, –7) lies on the circle, the distance of the centre from this point is the radius of the circle. Therefore, we get,

$$r=\sqrt{(4+2)^2+(-3+7)^2}=\sqrt{52}$$

Hence, the equation of the circle becomes

$$(x-4)^2+(y+3)^2=52$$
$$x^2+y^2-8x+6y-27=0$$

Illustration

A circle has radius 3 units and its centre lies on the line y = x – 1. Find the equation of the circle is it passes through (7, 3).

Solution

Let the centre of the circle be (h, k). Since the center lies on $y = x - 1$, we get

$$k = h - 1 \qquad \ldots \text{(i)}$$

Since the circle passes through the point (7, 3), therefore the distance of the centre from this point is the radius r of the circle. We have

$$r=\sqrt{(h-7)^2+(k-3)^2}$$

or $\quad 3=\sqrt{(h-7)^2+(h-1-3)^2}$

$\Rightarrow \quad 9=(h-7)^2+(h-4)^2$

$\Rightarrow \quad h^2-11h+28=0$

or $\quad (h-7)(h-4)=0$

or $\quad h=7$ and $h=4$

For $h = 7$, we get $k = 6$ from (1)

And for $h = 4$, we get $k = 3$ from (1).

Hence there are two circle which satisfy the given conditions. They are

$$(x-7)^2+(y-6)^2=9 \text{ or } x^2+y^2-14x+12y+76=0$$
$$(x-4)^2+(y-3)^2=9 \text{ or } x^2+y^2-8x-6y+16=0$$

Illustration

Find the area of an equilateral Δ inscribed in the circle

$$x^2+y^2+2gx+2fy+c=0$$

Solution

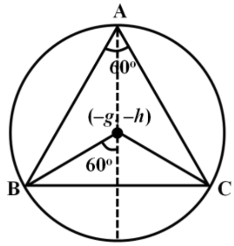
Given circle is

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{... (i)}$$

Let O be the centre and ABC be an equilateral triangle inscribed in the circle (1).

$O \equiv (-g, -f)$

and $OA = OB = OC = \sqrt{g^2 + f^2 - c}$ ... (ii)



In ΔOBM, $\sin 60^o = \frac{BM}{OB}$

$\Rightarrow$ $BM = OM \sin 60^o = (OB)\frac{\sqrt{3}}{2}$

$\therefore$ $BC = 2BM = \frac{\sqrt{3}}{2}\,(BC)^2$

$\therefore$ Area of $\Delta ABC = \frac{\sqrt{3}}{2}(OB)^2$

$= \frac{\sqrt{3}}{4}3(OB)^2 = \frac{3\sqrt{3}}{4}(g^2 + f^2 - c)$ sq. units

Illustration

Find the parametric form of the equation of circle. $x^2 + y^2 + Px + Py = 0.$

Solution

Equation of the circle can be re-written n the form

$$\left(x + \frac{P}{2}\right)^2 + \left(y + \frac{P}{2}\right)^2 = \frac{P^2}{2}$$

Therefore, the parametric form of the equation of the given circle is

$$x = -\frac{P}{2} + \frac{P}{\sqrt{2}}\cos\theta = \frac{P}{2}\left(-1 + \sqrt{2}\cos\theta\right) \quad x = -\frac{P}{2} + \frac{P}{\sqrt{2}}\sin\theta = \frac{P}{2}\left(-1 + \sqrt{2}\sin\theta\right)$$

Where $0 \le \theta < 2\pi$.

Illustration

If the parametric form of the circle is given by

(i) $x = -4 + 5\cos\theta$, $y = -3 + 5\sin\theta$ $\Rightarrow$ $(x + 4)^2 + (y + 3)^2 = 25$

(ii) $x = a\cos\alpha + b\sin\alpha$, $y = a\sin\alpha + b\cos\alpha$ $\Rightarrow$ $x^2 + y^2 = a^2 + b^2$

Solution

(i) The given equations are

$x = -4 + 5\cos\theta$ and $y = -3 + 5\sin\theta$

or $(x + 4) = 5\cos\theta$ ... (i)

and $(y + 3) = 5\sin\theta$ ... (ii)

Squaring and adding (1) and (2)

$(x + 4)^2 + (y + 3)^2 = 5^2$

or $(x + 4)^2 + (y + 3)^2 = 25$

(ii) The given equations are

$x = a\cos\alpha + b\sin\alpha$ ... (i)

$y = a\sin\alpha - b\cos\alpha$ ... (ii)

Squaring and adding (1) and (2), then

$x^2 + y^2 = (a\cos\alpha + b\sin\alpha)^2 + (a\sin\alpha - b\cos\alpha)^2$

$\Rightarrow$ $x^2 + y^2 = a^2 + b^2$.

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