## Chapter

## Pair of straight line

## Day - 1

## 1. Pair of Straight Line

Let $a_{1} x+b_{1} y+c_{1}=0$, and $a_{2} x+b_{2} y+c_{2}=0$ are the equation of two straight line.
Then their combined equation is known as Pair of Straight Line.

$$
\left(a_{1} x+b_{1} y+c_{1}\right)\left(a_{2} x+b_{2} y+c_{2}\right)=0
$$

Note:- First make RHS of two lines equal to zero and then multiply the equations.

## Illustration

Find the joint equation of lines.
$y=x$ and $y=-x$.

## Solution

First make RHS $=0$
i.e., $\quad y-x=0$
and $\quad y+x=0$
So their combined equation is $(y-x) .(y+x)=0$
Note:- To find the separate equations of two lines when their joint equation is given first of all make RHS equal to zero and resolve LHS into two linear factors or use Shri Dharacharya Method.

## Illustration

Find the separate equation of lines represented by the equation

## Solution

$$
\begin{array}{r}
x^{2}-6 x y+8 y^{2}=0 \\
(x-4 y)(x-2 y)=0
\end{array}
$$

so

$$
x-4 y=0 \text { and } x-2 y=0
$$

or

$$
\begin{aligned}
& x=\frac{6 y \pm \sqrt{(6 y)^{2}-4 \times 8 y^{2}}}{2} \\
& x=3 y \pm y \sqrt{9-8} \\
& x=3 y \pm y
\end{aligned}
$$

### 1.1 Homogeneous Equation in Two Variables

An equation of the form

$$
a_{0} y^{n}+a_{1} y^{n-1} x+a_{2} y^{n-2} x^{2}+\ldots+a_{n} x_{n}=0
$$

In which sum of powers of x and y in every form is the same, is called an homogeneous equation.
Divided by $x^{n}$ we get,

$$
a_{0}\left(\frac{y}{x}\right)^{n}+a_{1}\left(\frac{y}{x}\right)^{n-1} x+a_{2}\left(\frac{y}{x}\right)^{n-2} x^{2}+\ldots+a_{n}=0
$$

Above is an equation of $\mathrm{n}^{\text {th }}$ degree in $\frac{y}{x}$.
let the roots be $m_{1}, m_{2}, m_{3} \ldots m_{\mathrm{n}}$

Then
$a_{0}\left(\frac{y}{x}-m_{1}\right)\left(\frac{y}{x}-m_{2}\right) \ldots \ldots .\left(\frac{y}{x}-m_{n}\right)=0$
or $\quad y-m_{1} x=0, \quad y-m_{2} x=0 \ldots \ldots \ldots y-m_{n} x=0$
Since $a x^{2}+2 h x y+b y^{2}=0$ is a homogenous equation of second degree
so

$$
b\left(\frac{y}{x}\right)^{2}+2 h\left(\frac{y}{x}\right)+a=0
$$

so two equations are $\mathrm{y}=\mathrm{m} 1 \mathrm{x}$ and $\mathrm{y}=\mathrm{m} 2 \mathrm{x}$ are two straight line which pass through origin.
(i) The lines are real and distinct if $h^{2}-a b>0$
(ii) The lines are coincident if $h^{2}-a b=0$
(iii) The lines are imaginary if $h^{2}-a b<0$

### 1.2 Two Very Useful Identities

$$
a x^{2}+2 h x y+b y^{2}=y\left(y-m_{1} x\right)\left(y-m_{2} x\right)
$$

Comparing

$$
m_{1}+m_{2}=-\frac{2 h}{b}, \quad m_{1} m_{2}=\frac{a}{b}
$$

## Illustration

Find the condition that the slope of one of the lines represented by $a x^{2}+2 h x y+b y^{2}=0$ should be n times the slope of the other

$$
m_{1}+m_{2}=-\frac{2 h}{b}
$$

## Solution

Let the lines represented by

$$
\begin{align*}
a x^{2}+2 h x y+b y^{2} & =0 \text { are } y=m_{1} x \text { and } y=m_{2} x \\
m_{1}+m_{2} & =-\frac{2 h}{b}  \tag{i}\\
m_{1} m_{2} & =\frac{a}{b}  \tag{ii}\\
m_{2} & =n m_{1} \\
m_{1}+n m_{2} & =-\frac{2 h}{b} \\
m_{1} & =-\frac{2 h}{b(1+n)}  \tag{iii}\\
m_{1}\left(n m_{1}\right) & =\frac{a}{b} \\
n\left\{-\frac{2 h}{b(1+n)}\right\}^{2} & =\frac{a}{b} \\
\frac{2 n h^{2}}{b^{2}\left(1+n^{2}\right)} & =\frac{a}{b} \\
4 n h^{2} & =a b(1+n)^{2}
\end{align*}
$$

## Illustration

Show that the area of the triangle formed by the lines $a x^{2}+2 h x y+b y^{2}=0$ and $l x+m y+n=0$ is

$$
\frac{n^{2} \sqrt{h^{2}-a b}}{\left|a m^{2}-2 h l m+b l^{2}\right|}
$$

Solution

$$
\begin{array}{r}
A\left(-\frac{n}{l+m m_{1}},-\frac{n m_{1}}{l+m m_{1}}\right) \text { and } B\left(-\frac{n}{l+m m_{2}},-\frac{n m_{2}}{l+m m_{2}}\right) \\
\text { Area }=\frac{1}{2}\left|\left(-\frac{n}{l+m m_{1}}\right)\left(-\frac{n m_{2}}{l+m m_{2}}\right)-\left(-\frac{n}{l+m m_{2}}\right)\left(-\frac{n m_{1}}{l+m m_{1}}\right)\right|
\end{array}
$$

$$
\begin{aligned}
& =\frac{1}{2}\left|\frac{n^{2}\left(m_{2}-m_{1}\right)}{l^{2}+l m\left(m_{1}+m_{2}\right)+m^{2} m_{1 m_{2}}}\right| \\
& =\frac{1}{2}\left|\frac{n^{2} \sqrt{\left(m_{1}+m_{2}\right)^{2}-4 m_{1} m_{2}}}{l^{2}+l m\left(m_{1}+m_{2}\right)+m^{2} m_{1} m_{2}}\right| \\
& =\frac{1}{2}\left|\frac{n^{2} \sqrt{\left(\frac{4 h^{2} 4 a}{b^{2} b}\right)}}{l^{2}-\frac{2 h l m}{b}+\frac{m^{2} a}{b}}\right| \\
& =\frac{n^{2} \sqrt{h^{2}-a b}}{\left|a m^{2}-2 h l m+b l^{2}\right|}
\end{aligned}
$$



### 1.3 Angle between the pair of straight line $y=m_{2} x$ and $a x^{2}+2 h x y+b y^{2}=0$

$$
\tan \theta=\left\{\frac{2 \sqrt{h^{2}-a b}}{|a+b|}\right\}
$$

Let $y=m_{1} x$ and $y=m_{2} x$ be the line represented by

$$
a x^{2}+2 h x y+b y^{2}=0
$$

Then $\quad m_{1}+m_{2}=-\frac{2 h}{b}, \quad m_{1} m_{2}=\frac{a}{b}$
Since $\theta$ be the angle between the lines $y=m_{1} x$ and $y=m_{2} x$
Then $\tan \theta=\left|\frac{m_{1}-m_{2}}{1+m_{1} m_{2}}\right|$

$$
\begin{aligned}
& =\frac{\sqrt{\left(m_{1}+m_{2}\right)^{2}-4 m_{1} m_{2}}}{\left|1+m_{1} m_{2}\right|}=\frac{\sqrt{\left(-\frac{2 h}{b}\right)-4\left(\frac{a}{b}\right)}}{\left|1+\frac{a}{b}\right|} \\
\theta & =\tan ^{-1}\left\{\frac{2 \sqrt{h^{2}-a b}}{|a+b|}\right\}
\end{aligned}
$$

(i) Two lines are perpendicular if $(a+b)=0 \quad b=-a$

Hence perpendicular lines $a x^{2}+2 h x y+b y^{2}=0$
Coefficient of $x^{2}+$ coefficient of $y^{2}=0$
(ii) Pairs of lines perpendicular to the lines represented by $a x^{2}+2 h x y+b y^{2}=0$ and through origin $m_{1}+m_{2}=-\frac{2 h}{m}, \quad m_{1} m_{2}=\frac{a}{b}$
$\mathrm{y}=\mathrm{m}_{1} \mathrm{x}, \mathrm{y}=\mathrm{m}_{2} \mathrm{x}$
perpendicular to $y=-\frac{1}{m_{1}} x \quad y=\frac{1}{m_{2}} x$
For perpendicular lines inter charge the coefficient of $x 2$ and $y 2$ and change sigh of $x y$.
(iii) Condition for the lines to be coincident $\theta=0$ or $\pi$

$$
h^{2}-a b=0 \Rightarrow h^{2}=a b
$$

if $h^{2}=a b$ then $a x^{2}+2 h x y+b y^{2}$ is a perfect square.

## Illustration

Show that the straight lines $x^{2}+4 x y+y^{2}=0$ and line $x-y=4$ form an equilateral triangle.

## Solution

$$
\begin{aligned}
\frac{x}{y} & =\frac{-4 \pm \sqrt{4^{2}-4}}{2}=-2 \pm \sqrt{3} \\
y & =(-2 \pm \sqrt{3}) x \\
O P & =y+(2-\sqrt{3}) x=0
\end{aligned}
$$

$$
O Q=y+(2+\sqrt{3}) x=0
$$

Slope of $P Q=1$, slope of $O P=-(2-\sqrt{3})$ if $\angle P O Q=\alpha$

$$
\begin{aligned}
\tan \alpha & =\left|\frac{1-(-2+\sqrt{3})}{1-2+\sqrt{3}}\right|=\sqrt{3} \\
\alpha & =60^{\circ}
\end{aligned}
$$

## Illustration



Show that the condition that two of the three straight lines represent by $a x^{3}+b x^{2} y+c x y^{2}+d y^{3}=0$ may be at right angle if $a^{2}+a c+b d+d^{2}=0$.

## Solution

Let $y=m_{1} x, y=m_{2} x$ be the lines represented by the equation $a x^{3}+b x^{2} y+c x y^{2}+d y^{3}=0$. Then
$a x^{3}+b x^{2} y+c x y^{2}+d y^{3}=d\left[\left(y-m_{1} x\right)\left(y-m_{2} x\right)\left(y-m_{3} x\right)\right]$
On equating the coefficients of $x^{3}, x^{2} y$ and $x y^{2}$ on both sides, we get

$$
\begin{align*}
m_{1}+m_{2}+m_{3} & =-\frac{c}{d}, \\
m_{1} m_{2} m_{3} & =-\frac{a}{d} \tag{ii}
\end{align*}
$$

Let the perpendicular lines be $y=m_{1} x, y=m_{2} x$, then

$$
\begin{align*}
m_{1} m_{2} & =-1 \\
m_{3} & =\frac{a}{b} \tag{i}
\end{align*}
$$

Putting $y=m_{3} x$ in equation (i), then

$$
\begin{aligned}
a x^{3}+b m_{3} x^{3}+c m_{3}^{2} x^{3}+d m_{3}^{3} x^{3} & =0 \\
d m_{3}^{3}+c m_{3}^{2}+b m_{3}+a & =0 \\
d\left(\frac{a}{d}\right)^{3}+c\left(\frac{a}{d}\right)^{2}+b\left(\frac{a}{d}\right)+a & =0 \\
a^{2}+a c+b d+d^{2} & =0
\end{aligned}
$$

### 1.4 Bisectors of the Angle Between the Lines Given by a Homogeneous Equation.

$a x^{2}+2 h x y+b y^{2}=0 \Rightarrow$ Angle Bisector $\frac{x^{2}-y^{2}}{a-b}=\frac{x y}{h}$
or $\quad h x^{2}-(a-b) x y-h x y-h y^{2}=0$
i.e., coefficient of $x^{2}+$ coefficient of $y^{2}=0$

Hence the bisectors of the angle between the lines are always perpendicular to each other.
$>\quad$ If $a=b$ bisector $x^{2}-y^{2}=0$
i.e., $x-y=0$ or $x+y=0$
> If $h=0$ bisector are $x y=0$
i.e., $x=0, y=0$

## Illustration

Find the equation of the bisectors of the angle between the line represented by $3 x 2-5 x y+4 y^{2}=0$.

## Solution

Given equation is

$$
\begin{equation*}
3 x^{2}-5 x y+4 y^{2}=0 \tag{i}
\end{equation*}
$$

Comparing it with the equation

$$
\begin{equation*}
a x^{2}+2 h x y+b y^{2}=0 \tag{ii}
\end{equation*}
$$

Then $a=3, h=-5 / 2, b=4$
Hence equation of bisectors of the angle between the pair of the lines (i) is
$\frac{x^{2}-y^{2}}{3-4}=\frac{x y}{-\frac{5}{2}}$
$\Rightarrow \quad \frac{x^{2}-y^{2}}{-1}=\frac{2 x y}{-5}$
$\Rightarrow \quad 5 x^{2}-2 x y-5 y^{2}=0$.

## Illustration

Show that the line $\mathrm{y}=\mathrm{mx}$ bisects the angle between the lines $a x^{2}-2 h x y+b y^{2}=0$ if
$h\left(1-m^{2}\right)+m(a-b)=0$

## Solution

Equation of pair of bisectors of angles between lines $a x^{2}-2 h x y+b y^{2}=0$ is

$$
\begin{array}{cc}
\frac{x^{2}-y^{2}}{a-b}=\frac{x y}{-h} \\
\Rightarrow & -h\left(x^{2}-y^{2}\right)=(a-b) x y \tag{i}
\end{array}
$$

but $y=m x$ is one of these lines, then it will satisfy it. Substituting $y=m x$ in (i)

$$
-h\left(x^{2}-m^{2} x^{2}\right)=(a-b) x \cdot m x
$$

Dividing by $x^{2}$,

$$
h\left(1-m^{2}\right)+m(a-b)=0
$$

## Illustration

If pairs of straight line $x^{2}-2 p x y-y^{2}=0$ and $x^{2}-2 q x y-y^{2}=0$ be such that each pair bisects the angle between the other pair. Prove that $p q=-1$.

## Solution

According to the question, the equation of the bisectors of the angle between the lines

$$
\begin{array}{ll} 
& x^{2}-2 p x y-y^{2}=0 \\
\text { or } & x^{2}-2 p x y-y^{2}=0 \tag{ii}
\end{array}
$$

$\therefore$ The equation of bisectors of the angle between the lines (i) is

$$
\begin{gather*}
\frac{x^{2}-y^{2}}{1-(-1)}=\frac{x y}{-p} \\
\Rightarrow \quad-p x^{2}-2 x y+p y^{2}=0 \tag{iii}
\end{gather*}
$$

Since (ii) and (iii) are identical, comparing (ii) and (iii), we get

$$
\begin{array}{ll} 
& \frac{1}{-p}=\frac{-2 q}{2}=\frac{-1}{p} \\
\Rightarrow \quad & p q=-1
\end{array}
$$

## Online Question Practice

