Chapter 10 Linear Programming

Day – 1

Introduction

The term ‘programming’ means planning and it refers to a particular plane of action from amongst several alternatives for maximizing profit or minimizing cost etc. Programming problems deal with determining optimal allocation of limited resources to meet the given objectives, such as least cost, maximum profit, highest margin or least time, when resources have alternative uses.

The term ‘Linear means that all inequations or equations used and the function to be maximized or minimized are linear. That is why linear programming deals with that class of problems for which all relations among the variables involved are linear.

Formally, linear programming deals with the optimization (maximization or minimization) of a linear function of a number of variables subject to a number of conditions on the variables, in the form of linear inequations or equations in variables involved.

In this chapter, we shall discus mathematical formulation of linear programming problems that arise in trade, industry, commerce and military operations. We shall also discuss some elementary techniques to solve linear programming problems in two variables only.

Linear Programming Problems

In this section, we shall discuss the general form of a linear programming problem. To give the general description of a linear programming problem, let us consider the following problem:

Suppose that a furniture dealer makes two products viz.chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 4 hours on machine A and 2 hours on machine B. There are 16 hours of time per day available on machine A and 20 hours on machine B. Profits gained by the manufacturer from a chair and a table are Rs. 30 and Rs. 50 respectively. The manufacturer is willing to know that daily product of each of the two products to maximize his profit.

The above data can be put in the following tabular form:

Item	Chair	Table	Maximum available time
Machine A	2 hrs	4 hrs	16 hrs
Machine B	6 hrs	2 hrs	20 hrs
Profit (in Rs)	Rs 30	Rs 50	

To maximize his profit, suppose that the manufacture produces x chairs and y tables per day. It is given that chair requires 2 hours on machine A and a table requires 4 hours on machine A. Hence, the total time taken by machine A to produce x chairs and y tables is $2x + 4y$. This must be less than or equal to the total hours available on machine A. Hence, $2x + 4y \leq 16$. Similarly, for machine B, we have

$$6x + 2y \leq 20$$

The total profit for x chairs and y tables is $30x + 50y$. Since the number of chairs and tables is never negative. Therefore, $x \geq 0$ and $y \geq 0$.

Thus, we have to maximize

Profit = $30x + 50y$

Subject to the constraints

$$2x + 4y \leq 16$$
$$6x + 2y \leq 20$$
$$x \geq 0,\ y \geq 0$$

Out of all the points (x, y) in the solution set of the above linear constraints, the manufacturer has to choose that point, or those points for which the profit $30x + 50y$ has the maximum value. In the above discussion if a chair Rs 250 and a table costs Rs 300 then total cost of producing x chairs and y tables is $250x + 300y$. Now, the manufacturer will be interested to choose that point, or those points, in the solution set of the above linear constraints for which the cost $250x + 300y$ has the minimum value.

The two situations discussed above give the description of a type of linear programming problems. In the above discussion, the profit function = $30x + 50y$ or the cost function $250x + 300y$ is known as the objective function. The inequations $2x + 4y \leq 16,\ 6x + 2y \leq 20$ are known as the constraints and $x \geq 0,\ y \geq 0$ are known as the non-negativity restrictions.

The general mathematical description of a linear programming problem (LPP) is given below:

Optimize Z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$ (objective function)

Subject to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n\ (\leq, =, \geq)b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n(\leq, =, \geq)b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \text{(constraints)}$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots\ldots + a_{mn}x_n(\leq, =, \geq)b_m$$
$$x_1, x_2, x_3, \ldots\ldots, x_n \geq 0 \qquad \text{(non-negativity restrictions)}$$

Where all a_{ij}'s, b_i's and c_j's are constants and x_j's are variables.

The above linear programming problem may also be written in the matrix form as follows:

Optimize (maximize or minimize) $Z = [c_1 c_2 \ldots\ldots c_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Subject to

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\{\leq, =, \geq\}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$x_1, x_2, x_3, \dots \dots x_n \geq 0$

Optimize (Maximize or Minimize)

$Z = CX$

Subject to

$AX\ (\leq, =, \geq) B$

$X \geq 0,$

Where $C = [c_1\ c_2\ \dots . c_n], X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$

Some Definitions

In this section, we shall formally define various terms used in a linear programming problem. As discussed in the pervious section, the general form of a linear programming problem is

Optimize (Maximize or Minimize) $Z = c_1x_1 + c_2x_2 + \dots .+ c_nx_n$

Subject to

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \dots \dots + a_{1n}x_n\ (\leq, =, \geq) b_1 \\ &a_{21}x_1 + a_{22}x_2 + \dots \dots + a_{2n}x_n (\leq, =, \geq) b_2 \\ &\quad\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ &a_{m1}x_1 + a_{m2}x_2 + \dots \dots + a_{mn}x_n (\leq, =, \geq) b_m \\ &x_1, x_2, \dots \dots, x_n \geq 0 \end{aligned}$$

The definitions of various terms related to the LPP are as follows:

Objective Function

If $c_1, c_2, \dots . c_n$ are constants and $x_1, x_2, \dots x_n$ are variables, then the linear function $Z = c_1x_1 + c_2x_2 + \dots . c_nx_n$ which is to be maximized or minimized is called the objective function.

The objective function describes the primary purpose of the formulation of a linear programming problem and it is always non-negative. In business applications, the profit function which is to maximized or the cost function which is to be minimized is called the objective function.

Constraints

The inequations or equations in the variables of a LPP which describe the conditions under which the optimization (maximization or minimization) is to be accomplished are called constraints. In the constraints given in the general form of a LPP there may be any one of the three sings $\leq, =, \geq$. Inequations in the form of greater than (or less than) indicate that the total use of the resources must be more than (or less than) the specified amount whereas equations in the constraints indicate that the resources described are to be fully used.

Non-Negativity Restrications

These are the constraints which describe that the variables involved in a LPP are non-negative.

Mathematical Formulation of Linear Programming Problems

In the previous section, we have introduced the general form of a linear programming problem (LPP). In this section, we shall discuss the formulation of linear programming problems. Problem formulation is the process of transforming the verbal description of a decision problem into a mathematical form. There is not any set procedure to formulate linear programming problems, the following algorithm will be helpful in the formulation of linear programming problems. The following examples will illustrate the formulation of linear programming problems in various situations.

Illustration

A toy company manufactures two types of doll; a basic version doll A and deluxe version doll B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2,000 per day if it produces only the basic version. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes profit of Rs 3 and Rs 5 per doll respectively on doll A and B; how many of each should be produced per day in order to maximize profit?

Solution

Let x dolls of type A and y dolls of type B be produced per day. Then,

Total profit = $3x + 5y$.

Since each doll of type B takes twice as long to produce as one of type A, therefore total time taken to produce x dolls of type And y dolls of type B is $x + 2y$. But the company has time to make a maximum of 2000 dolls per day

∴ $x + 2y \leq 2000$

Since plastic is available to produce 1500 dolls only.

∴ $x + y \leq 1500$

Also fancy dress is available for 600 dolls per day only.

∴ $y \leq 600$

Since the number of dolls cannot be negative. Therefore,

$$x \geq 0,\ y \geq 0$$

Hence, the linear programming problem for the given problem is as follows:

Maximize Z = $3x + 5y$

Subject to the constraints

$$x + 2y \leq 2000$$
$$x + y \leq 1500$$
$$y \leq 600$$
$$x \geq 0,\ y \geq 0.$$

Illustration

A manufacturer of line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20,000 bottles of A and 40,000 bottles of B but there are only 45,000 bottles into which either of the medicines can be put. Further more, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes one hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is Rs 8 per bottle for A and Rs 7 per bottle for B. Formulate this problem as a medicine B.

Solution

Suppose the manufacture produces x bottles of medicines A and y bottles of medicine B.

Since the profit is Rs 8 per bottle for A and Rs 7 per bottle for B. So, total profit in producing x bottles of medicine A and y bottles of medicine B is Rs $(8x + 7y)$.

Let Z denote the total profit. Then,

$$Z = 8x + 7y$$

Since 1000 bottles of medicine A are prepared in 3 hours. So,

Time required to prepare x bottles of medicine $A = \frac{3x}{1000}$ hours.

It is given that 1000 bottles of medicine B are prepared in 1 hour.

∴ Time required to prepare y bottles of medicine $B = \frac{y}{1000}$ hour.

Thus, total time required to prepare x bottles of medicine A and y bottles of medicine B is $\frac{3x}{1000} + \frac{y}{1000}$ hours. But, the total time available for this operation is 66 hours.

∴ $$\frac{3x}{1000} + \frac{y}{1000} \le 66$$

⇒ $$3x + y \le 66000$$

Since there are only 45,000 bottles into which the medicines can be put.

∴ $$x + y \le 45{,}000$$

Since the number of bottles cannot be negative. Therefore, $x \ge 0,\ y \ge 0$.

Hence, the mathematical formulation of the given LPP is as follows:

Maximize $Z = 8x + 7y$

Subject to

$$3x + y \le 66{,}000$$
$$x + y \le 45{,}000$$
$$x \le 20{,}000$$
$$y \le 40{,}000$$
$$x > 0,\ y \ge 0.$$

Illustration

A company makes two kinds of leather belts, A and B. Belt A is high quality belt, and B is lower quality. The respective profits are Rs 4 and Rs 3 per belt. Each belt of type A requires twice as much time as belt of type B, and if all belts were of type B, the company could make 1000 belts per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle, and only 400 buckles per day are available. There are only 700 buckles available for belt B. What should be the daily production of each type of belt? Formulate the problem as a LPP.

Solution

Suppose the company makes per day *x* belts of type A and *y* belts of type B.

∴ Profit = $4x + 3y$

Let Z denote the profit. Then, $Z = 4x + 3y$ and it is to be maximized.

It is given that 1000 belts of type B can be made per day and each belt of type A requires twice as much time as a belt of type B. So, 500 belts of type A can be made in a day.

So, total time taken in preparing x belts of type A and y belts of B is $\left(\frac{x}{500} + \frac{y}{1000}\right)$. But the company is making x belts of type A and y belts of type B in a day.

∴ $\frac{x}{500} + \frac{y}{1000} \leq 1 \Rightarrow 2x + y \leq 1000.$

Since the supply of leather is sufficient for only 800 belts per day.

∴ $x + y \leq 800.$

It is given that only 400 fancy buckles for type A and 700 buckles for type B are available per day.

∴ $x \leq 400, \; y \leq 700$

Finally, the number of belts cannot be negative.

∴ $x \geq 0$ and $y \geq 0$

Thus, the mathematical formulation of the given LPP is as follows:

Maximize $Z = 4x + 3y$

Subject to

$$2x + y \leq 1000$$
$$x + y \leq 800$$
$$x \leq 400$$
$$y \leq 700$$
$$x \geq 0, \; y \geq 0.$$

Illustration

The objective of a diet problem is to ascertain the quantities of certain foods that should be eaten to meet certain nutritional requirement at a minimum cost. The consideration is limited to milk, beaf and eggs, and to vitamins A, B, C. The number of milligrams of each of these vitamins contained within a unit of each food is given below:

Vitamin	Litre of milk	Kg of beaf	Dozen of eggs	Minimum daily requirements
A	1	1	10	1 mg

B C	100 10	10 100	10 10	50 mg 10mg
Cost	Rs 1.00	Rs 1.10	Re 0.50	

What is the linear programming formulation for this problem?

Solution

Let the daily diet consists of x litres of milk, y kgs of beaf and z dozens of eggs. Then,

Total cost per day = Rs (x + 1.10y + 0.50z).

Let Z denote the total cost in Rs. Then,

$$Z = x + 1.10y + 0.50z$$

Total amount of vitamin A in the daily diet is

(x + y + 10z)mg

But the minimum requirement is 1 mg of vitamin A.

∴ $x + y + 10z \geq 1$

Similarly, total amounts of vitamins B and C in the daily diet are (100x + 10y + 10z)mg and (10x + 100y + 10z)mg respectively and their minimum requirements are of 50 mg and 10 mg respectively.

∴ $100x + 10y + 10z \geq 50$

$10x + 100y + 10z \geq 10$

Finally, the quantity of milk, kgs of beaf and dozens of eggs can not assume negative values.

∴ $x \geq 0,\ y \geq 0,\ z \geq 0$

Hence, the mathematical formulation of the given LPP is

Minimize Z = x + 1.10y + 0.50z

Subject to

$x + y + 10z \geq 1$

$100x + 10y + 10z \geq 50$

$10x + 100y + 10z \geq 50$

$x \geq 0,\ y \geq 0,\ z \geq 0.$