Chapter 8 Matrices

Day – 1

Definition

A set of mn numbers arranged in the form of an ordered set of m rows and n columns is called m× n matrix (to be read as m by n matrix)

Thus m × n matrix A is written as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Or $A = [a_{ij}]$; i = 1,2,…m and j = 1,2,…n

Or $A = [a_{ij}]_{m\times n}$

Where aij represents the element at the intersection of ith row and jth column.

In case the order of a matrix is established or known then we shall simply write

$$A = [a_{ij}] \text{of type } m \times n.$$

Various Types of Matrix

(a) Square matrix:-

A matrix in which the number of rows is equal to the number of columns is called a square Matrix. Thus m × n matrix A will be a square matrix if m = n ad it will be termed as a square matrix of order n or n – rowed square matrix.

(b) Diagonal Elements:-

In a square matrix all those element a_{ij} for which i = j i.e. all those elements which occur in the same row and same column namely a_{11}, a_{22}, a_{33} are called the diagonal elements and the line along which they lie is called the principle diagonal. Also the sum of the diagonal elements of a square matrix A is called trace of A.

$\text{i.e. } a_{11} + a_{22} + a_{33} + \dots = \text{Trace of A}$

In general a_{11} , a_{22} ….anm are the diagonal elements of n – rowed square matrix and

$$a_{11} + a_{22} + \dots a_{nn} = \text{Trace of A.}$$

(c) Diagonal Matrix:-

A square matrix A is said to be a diagonal matrix if all its non – diagonal elements be zero.

$$\text{Thus} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{vmatrix}$$

Above are diagonal matrix of the type 3 × 3 These are in short written as

Diag[1,4,8] or Diag $[d_1, d_2, d_3]$

(d) Scalar Matrix:-

A diagonal matrix whose all the diagonal elements are equal is called a scalar matrix.

Thus $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ or $\begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix}$

Are both scalar matrixes of type3×3.

In general for a scalar matrix.

$a_{ij} = 0 \quad \text{for } i \neq j \text{ and } a_{ij} = d \quad \text{for } i = j$

(e)Unit Matrix:-

A square matrix A all of whose non –diagonal elements are zero(i.e.it is a diagonal matrix) and also all the diagonal elements are unity is called a unit matrix or an identity matrix.

Thus $\begin{bmatrix} 1 & o & o \\ o & 1 & o \\ o & o & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

are unit matrix of order 3 and 4 respectively.

In general for a unit matrix.

$a_{ij} = 0 \quad \text{for } i \neq j \text{ and } a_{ij} = 1 \text{ for } i = j.$

They are generally denoted by I_3, I_4 or I_n where 3,4,n denoted of the square matrix. In case the order be know then we may simply denote it by I.

(f) Zero Matrix or Null Matrix:-

Any m × n matrix in which all the element are zero is called a zero matrix or null matrix of the type m × n and is denoted by $O_{m \times n}$.

Thus $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

All the above are zero or null matrices of the type 3× 3, 3×3 and 2×4 respectively.

(g) Determinant of a square Matrix:-

if we have a square matrix having same number of rows and columns it will have $n \times n = n^2$ arrays of numbers. These n^2 numbers also determine a determinant having n rows and n columns and is denoted by Det A or |A|.

(h) Equality of Matrices:-

Two matrices $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$ are said to be equal and written as A = B if and only if they have the same order or are of the same type i.e., each has as many rows and columns as the order [In this case they are said to be comparable and also each element of one is equal to the corresponding element of the order i.e., $a_{ij} = b_{ij}$ for each pair of subscripts i and j where i = 1, 2 ,…m and j = 1,2,…n]

Hence we can say that two matrix are equal if and only if one is duplicate of the other.

(i) Sum of Matrices:-

Let A = [aij] and B [bij] be two matrices of the same type m × n. Then their sum (or difference) A+ B (or A - B) is defined as another matrix of the same type, say C= [cij] such that any element of C is the sum (or difference) of the corresponding elements of A and B.

$$\therefore \quad C = A \pm B = [a_{ij} \pm b_{ij}]$$

$$\text{e.g. } A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 7 & 3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$$

Hence both A and B are 2× 3 matrices.

$$\therefore \quad A + B = \begin{bmatrix} 1+7 & 2+3 & 4+2 \\ 0+5 & 5+1 & 3+9 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 6 \\ 5 & 6 & 12 \end{bmatrix}$$

$$\text{and } A - B = \begin{bmatrix} 1-7 & 2-3 & 4-2 \\ 0-5 & 5-1 & 3-9 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & 2 \\ -5 & 4 & -6 \end{bmatrix}$$

(j) Negative of a Matrix:-

If A be a given Matrix then – A is called the negative of matrix A and all its element are the corresponding elements of A multiplied by – 1.

$$\text{Thus if } A = \begin{bmatrix} 2 & 3 & -1 \\ 6 & -4 & 2 \end{bmatrix}$$

$$\text{Then } -A = \begin{bmatrix} -2 & -3 & 1 \\ -6 & 4 & -2 \end{bmatrix}$$

(k) Scalar Multiple of a Matrix:-

If A be a given matrix and k is any scalar number real or complex. [We call it scalar k to disintiguish it from matrix[k] which is 1×1 matrix] then by matrix kA = Ak is meant the matrix all of whose elements are k times of the corresponding elements of A.

$$\text{If } A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 4 \end{bmatrix}$$

$$\text{then } 3A = \begin{bmatrix} 3.2 & 3.3 & 3.1 \\ 3.5 & 3.2 & 3.4 \end{bmatrix}$$

$$\text{or } 3A = \begin{bmatrix} 6 & 9 & 3 \\ 15 & 6 & 12 \end{bmatrix}$$

$$\text{Similarly } -4A = \begin{bmatrix} -4.2 & -4.3 & -4.1 \\ -4.5 & -4.2 & -4.4 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & -12 & -4 \\ -20 & -8 & -16 \end{bmatrix}$$

Properties of Matrix Multiplication

(a) Multiplication of matrices is distributive with respect to addition of matrices.

$i.e., \quad A(B+C) = AB + AC.$

(b) Matrix multiplication is associative if conformability is assured.

$i.e., \quad A(BC) = (AB)c.$

(c) The multiplication of matrices is not always commutative. i.e., AB is not always equal to BA.

(d) Multiplication of a matrix A by a null matrix conformable with A for multiplication is a null matrix i.e., AO = O

(e) If AB = O then it does not necessarily mean that A = O or B = O or both are O as shown below.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

None of the matrix on the left is a null matrix whereas their product is a null matrix.

(f) Multiplication of matrix A by a unit matrix I: Let A be a m $\times$n matrix and I be a square unit matrix of order n. so that A and I are conformable for multiplication, then

$$AI_n = A.$$

Similarly for IA to exist I should be square unit matrix of order m and in that case I_m A =A

Illustration

If $A + B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ and $A - B = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$

then prove that $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -2 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution

Adding and subtracting we get 2A and 2B. Hence A and B are as given.

Illustration

If $2X - Y = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}$ and $2Y + X = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$ then find X and Y

Solution

Eliminate Y and Find X. Put for X and find Y.

$$X = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, Y = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$$

Illustration

If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove the following

(a) $(A_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$

(b) $A_\alpha A_\beta = A_{\alpha+\beta} = A_\beta A_\alpha$.

Solution

(a):-

$$(A_\alpha)^2 = A_\alpha A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{matrix} \cos^2\alpha - \sin^2\alpha & 2\sin\alpha\cos\alpha \\ -2\sin\alpha\cos\alpha & -\sin^2\alpha + \cos^2\alpha \end{matrix}$$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Similarly, $(A_\alpha)^3 = (A_\alpha)^2(A_\alpha)$

$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\alpha+\alpha) & \sin(2\alpha+\alpha) \\ -\sin(2\alpha+\alpha) & \cos(2\alpha+\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos 3\alpha & \sin 3\alpha \\ -\sin 3\alpha & \cos 3\alpha \end{bmatrix}$$

In the light of above let us assume that

$$(A_\alpha)^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

$\therefore \quad (A_\alpha)^{n+1} = (A_\alpha)^n A_\alpha$

$$= \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}\begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(n+1)\alpha & \sin(n+1)\alpha \\ -\sin(n+1)\alpha & \cos(n+1)\alpha \end{bmatrix}$$

Thus we observe that our assumption for $(A_\alpha)^n$ is true for n = n+1 and it was shown to be true for n = 2,3,.. and hence it is true universally.

(b):-

$$A_\alpha A_\beta = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = A_{\alpha+\beta}$$

Also $A_\beta . A_\alpha = A_{\alpha+\beta}$ which can be shown as above.

Illustration

If $A = \begin{bmatrix} 0 & -\tan\alpha/2 \\ \tan\alpha/2 & 0 \end{bmatrix}$ and I is a 2×2 unit matrix, then prove that

$$I + A = (I - A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Solution

$$\text{L.H.S. } I + A = \begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix}$$

as $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\text{R.H.S.} = \begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha + \sin\alpha\tan\alpha/2 & -\sin\alpha + \cos\alpha\tan\alpha/2 \\ \left(-\tan\frac{\alpha}{2}\right)\cos\alpha + \sin\alpha & (\tan\alpha/2)\sin\alpha + \cos\alpha \end{bmatrix}$$

Now changing tanα/2 into sin α/2/cos α/2 and applying the formula for $\sin(A \pm B)$ and $\cos(A \pm B)$

$$\text{R.H.S} = \frac{1}{\cos\alpha/2}\begin{bmatrix} \cos(\alpha - \alpha/2) & -\sin(\alpha - \alpha/2) \\ \sin(\alpha - \alpha/2) & \cos(\alpha - \alpha/2) \end{bmatrix}$$

$$= \frac{1}{\cos\alpha/2}\begin{bmatrix} \cos\alpha/2 & -\sin\alpha/2 \\ \sin\alpha/2 & \cos\alpha/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix} = I + A \text{ by } (1)$$