Chapter 7 Determinants

Day – 1

Definition

Consider the equations

$$a_1\text{x} + \text{b}_1\text{y} = 0, a_2\text{x} + \text{b}_{2\text{y}} = 0$$

These give $-\frac{a_1}{\text{b}_1} = \frac{\text{y}}{\text{x}} = -\frac{a_2}{\text{b}_2},$

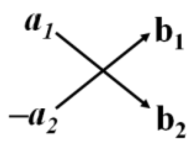
Hence $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ or $a_1 b_2 - a_2 b_1 = 0$

We shall express the above eliminant as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \qquad \text{... (A)}$$

We have suppressed the letter x and y to be eliminated and enclosed their coefficient as above in two parallel lines. The left hand member of (A) is called a determinant of second order and its value as we have seen is $a_1b_2 – a_2b_1$.

Aid to Memory



Similarly a determinant of 3rd order will consist of 3 rows and 3 columns enclosed in two verticals lines and is thus of the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \text{... (B)}$$

It can be seen that this determinant is the eliminant of *x*, *y*, *z* from the equations

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0$$

The value of determinant (B) is

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \qquad \text{... (1)}$$

Rule

a_1(determinant obtained by removing the row and column intersecting at a_1)

– b_1 (determinant obtained by removing the row and column intersecting at b_1)
+ c_1 (determinant obtained by removing the row and column intersecting at c_1)
Above is called expansion of the determinant w.r.t. first row.

Expansion With Respect to First Column

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \qquad \ldots(2)$$

If you compare (1) and (2) term by term you will observe that are same.

Properties

1. The value of determinant is not altered by changing rows into columns and columns into rows.

$$e.g.\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

2. If any two adjacent rows or two adjacent columns of a determinant are interchanged the determinant retains its absolute value but change its sing

$$e.g.\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Here we have interchanged 1^{st} and 2^{nd} rows and hence changed the sign.

3. If any line of a determinant Δ be passed over *p* parallel lines the resultant determinants is $(-1)^P$ Δ.

$$e.g.\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & u \\ x^2 & y^2 & z^2 & u^2 \\ x^3 & y^3 & z^3 & u^2 \end{vmatrix}$$

$$= (-1)^3\begin{vmatrix} x^3 & y^3 & z^3 & u^3 \\ 1 & 1 & 1 & 1 \\ x & y & z & u \\ x^2 & y^2 & z^2 & u^2 \end{vmatrix}$$

$$= (-1)^2\begin{vmatrix} x^2 & y^2 & z^2 & u^2 \\ 1 & 1 & 1 & 1 \\ x & y & z & u \\ x^3 & y^3 & z^3 & u^3 \end{vmatrix}$$

$$= (-1)^1 \begin{vmatrix} x & y & z & u \\ 1 & 1 & 1 & 1 \\ x^2 & y^2 & z^2 & u^2 \\ x^3 & y^3 & z^3 & u^3 \end{vmatrix}$$

In the first we have crossed fourth row over three parallel rows and hence $(-1)^3$ and in the second we have crossed 3rd row over two parallel rows and hence $(-1)^2$ and in the last we have crossed 2nd row over one parallel row and hence $(-1)^1$. Similarly is the rule for crossing any column over other columns.

4. If any two rows or two columns of a determinant are identical then the determinant vanishes. Thus

$$\begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix} = 0$$

5. If each constituent in any row or in any column be multiplied by the same factor then the determinant is multiplied by that factor

$$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We have taken p common from 1st column.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ qa_2 & qb_2 & qc_2 \\ ra_3 & rb_3 & rc_3 \end{vmatrix} = qr \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We have taken q and r from the 2nd and 3rd rows respectively.

6. If each constituent in any row or column consists of r terms then the determinant can be expressed as the sum of r determinants. Thus

$$\begin{vmatrix} a_1 + \alpha_1 & b_1 & c_1 \\ a_2 + \alpha_2 & b_2 & c_2 \\ a_3 + \alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & c_1 + \alpha_1 + \beta_1 \\ a_2 & b_2 & c_2 + \alpha_2 + \beta_2 \\ a_3 & b_3 & c_3 + \alpha_3 + \beta_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \alpha_1 \\ a_2 & b_2 & \alpha_2 \\ a_3 & b_3 & \alpha_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \beta_1 \\ a_2 & b_2 & \beta_2 \\ a_3 & b_3 & \beta_3 \end{vmatrix}$$

7. If from each constituent of a row (or column) of a determinant are added or subtracted the equi – multiples of the corresponding constituent of any other row (or column) the determinant remains unaltered.

$$\text{e.g. consider } \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now suppose we add to 1st column, p times the corresponding elements of 2nd column and subtract q times of the corresponding elements of 3rd column then the value of the determinant remains unaltered

Thus

$$\begin{vmatrix} a_1 + pb_1 - qc_1 & b_1 & c_1 \\ a_2 + pb_2 - qc_2 & b_2 & c_2 \\ a_3 + pb_3 - cq_3 & b_3 & c_3 \end{vmatrix} = \Delta$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p.0 - q.0 \qquad ...(4)$$

Illustration

Evaluate the determinant without expansion as far as possible.

$$\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$$

Solution

$$\Delta = \begin{vmatrix} 7.6 + 1 & 1 & 6 \\ 7.4 + 7 & 7 & 4 \\ 7.2 + 3 & 3 & 2 \end{vmatrix}$$

$$= 7 \begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$$

$$= 7.0 + 0 = 0$$

Both the determinants are zero because of identical columns.

Illustration

Evaluate the determinant without expansion as far as possible.

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Where ω is an imaginary cube root of unity.

Solution

We know that if ω is a cube root of unity then $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$ and ω 4 = $\omega^3 . \omega = \omega$ etc.

Hence applying $C_1 + C_2 + C_3$ we see that each element of the first column because $1 + \omega + \omega^2$ i.e. zero

∴ $\Delta = 0.$

Illustration

Evaluate the determinant without expansion as far as possible.

$$\begin{vmatrix} 1 & a & b + c \\ 1 & b & c + a \\ 1 & c & a + b \end{vmatrix}$$

Solution

Apply $C_3 + C_2$ etc., $\Delta = 0$

Illustration

Evaluate $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$

Solution

Multiply R_1, R_2, R_3 by a,b,c respectively and hence divided by abc.

$$\therefore \qquad \Delta = \frac{1}{abc}\begin{vmatrix} (abc)bc & abc & a(b+c) \\ (abc)ca & abc & b(c+a) \\ (abc)ab & abc & c(a+b) \end{vmatrix}$$

$$= \frac{(abc)^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ca & 1 & bc+ba \\ ab & 1 & ca+cb \end{vmatrix}$$

Apply $C_3 + C_1$ and take out $\sum ab$ and then C_2 and C_3 become identical.

$\therefore \qquad \Delta = 0.$

Illustration

Evaluate : $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

Solution

Answer = 0

$$\log_n m = \frac{\log m}{\log n}.$$

Hence each row becomes

$$\log x \log y \log z \text{ i.e. identical.}$$