Chapter 6 Binomial Theorem

Day – 1

1. Statement of Binomial Theorem for Positive Integral Index

$$(x+a)^n = x^n + {}^nC_1\ x^{n-1}\, a^1 + {}^nC_2\, x^{n-2}\, a^2 + \cdots$$
$$+\,{}^nC_r\, x^{n-r}\, a^r + \cdots + \cdots + {}^nC_{n-1}\, x\, a^{n-1} + {}^nC_n\, a^n$$

If the binomial be x – a then the terms in the expansion of $(x-a)^n$ will be alternaterly + ive and – ive.

2. General Term, T_{r+1}

${}^nC_r\, x^{n-r}.\, a^r$ for $(x+a)^n$

or ${}^nC_r\, x^{n-r}\, (-a)^r.$ for $(x-a)^n$

The index of x is $n-r$ and that of a is r *i.e.* sum of the indices of x and a in each term is same *i.e.*, n .

$$\left| {}^nC_r\ (\text{I})^{n-r}\ (\text{II})^r \right|$$

3. (a) Binomial Coefficients of Terms Equidistant From the Beginning and the are Equal

Since ${}^nC_r = {}^nC_{n-r},$

$\therefore$ ${}^nC_0 = 1 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1},$

${}^nC_2 = {}^nC_{n-2}$ etc.

(b) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(c) ${}^nC_r = {}^nC_p \implies$ either $r = p$ or $r + p = n$

(d) $-\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{n}{r}$

(e) ${}^nC_r$ is gratest when $r = \frac{n}{2}$, (n being even)

(f) ${}^nC_r$ is greatest when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$

(n being odd) and ${}^nC_{(n+1)/2} = {}^nC_{(n-1)/2}$

4. Number of Terms and Middle Term

The number of terms in the expansion of $(x+a)^n$ is $n+1$.

If $n = 6$ the number of terms will be 6 + 1 = 7 and the middle will only one *i.e.* 4^{th}

$i.e. \frac{6}{2} + 1 = 4.$

If n = 7 the number of terms will be 7 + 1 = 8 and in this case there will be 2 middle terms *i.e.* 4th and 5th

$$\frac{7+1}{2} = 4 \text{ and} \frac{7+3}{2} = 5.$$

Hence if n is even there will be only one middle term $i.e., \left(\frac{n}{2} + 1\right)$ th.

If n is odd then there will be two middle terms

$i.e., \left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+3}{2}\right)$ th.

5. Values of Binomial Coefficients

$${}^nC_0 = 1,\ {}^nC_1 = \frac{n!}{(n-1)!.1!} = n.$$

$${}^nC_2 = \frac{n!}{(n-2)!.2!} = \frac{n(n-1)}{2!},$$

$${}^nC_3 = \frac{n(n-1)(n-2)}{3!}$$

and in general.

$${}^nC_2 = \frac{n(n-1)(n-2)...(n-r+1)}{r!}$$

$$(1+x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + \cdots + {}^nC_r\, x^r + \cdots + {}^nC_n\, x^n.$$

6. Term Containing x^r will occur in T_{r+1} for $(1 + x)^n$ and it will be ${}^nC_r\, x^r$

7. $\frac{T_{r+1}}{T_r}$ for $(x + a)^n$

$$\frac{T_{r+1}}{T_r} = \frac{{}^nC_r\, x^{n-r}\, a^r}{{}^nC_{r-1}\, x^{n-r+1}.a^{r-1}}$$

$$= \frac{n!}{r!(n-r)!} \frac{(r-1)!(n-r+1)!}{n!} \frac{a}{x}$$

$$= \frac{n-r+1}{r} . \frac{a}{x} \qquad ...(1)$$

Greatest term in $(1 + x)^n$, n > 0

Evaluate $m = \left(\frac{x}{x+1}\right)(n+1)$

In case m is an integer, then T_m and T_{m+1} will be equal and both these will be numerically the greatest terms.

In case m is not an integer, then evaluate $[m]$ *i.e.* greatest integer, then $T_{[m]+1}$ will be the greatest term.

8. (i) Term Independent of *x* in the Expansion of $(x + a)^n$

Let T_{r+1} be the term independent of x. Equate to zero the index of z and you will find the value of r.

(ii) Terms Equidistant From the Beginning and End of the Binomial Expansion $(x+a)^n$

T_{r+1} from begining of $(x+a)^n = {}^nC_r x^{n-r} a^r$

T_{r+1} from end of $(x+a)^n$ is T_{r+1} from beginning of $(x+a)^n$ (binomial reversed) and is equal to ${}^nC_r a^{n-r} x^r$.

Illustration

Find the term independent of x in the expansion of $\left(3\text{x} - \frac{2}{\text{x}^2}\right)^{15}$

Solution

Let T_{r+1} be independent of x, *i. e.* index of x is zero.

$\left(3x - \frac{2}{x^2}\right)^{15}$

$T_{r+1} = {}^{15}C_r\ (3x)^{15-r}\left(-\frac{2}{x^2}\right)^r$

$= (-1)^r\ {}^{15}C_r\ 3^{15-r}.2^r.x^{15-r-2r}.$ $\quad \ldots(1)$

The index of x is 15 - 3r = 0 ∴ r = 5.

Hench the 6th term is the required term . putting

r = 5in (1), we get

$T_6 = (-1)^5\ {}^{15}C_5 3^{10} 2^5 x^0$

$= 3^{10} 2^5.\frac{15!}{5!(10)!}$

$= 3^{10} 2^5.\frac{15\times14\times13\times12\times11}{5.4.3.2.1}$

$= (3003)\ 3^{10} 2^5.$

Illustration

Find the coefficient of x^{10} and x^9 in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{20}$

Solution

$T_{r+1} = {}^{20}C_r (2x^2)^{20-r}\left(-\frac{1}{x}\right)^r$

$= (-1)^r\ {}^{20}C_r\ 2^{20-r}\ x^{40-2r-r}$

∴ 40 - 3r = 10 or 9

∴ 3r = 30 or 31.

∴ r = 10, the other value does not give integral value of r so that there will be no term of x^9.

Putting r = 10,

$T_{11} = (-1)^{10}\ {}^{20}C_{10}\ 2^{20-10}\ x^{40-30}$

$= \frac{(20)!}{(10)!(10)!} 2^{10} x^{10}$

Hence the required coefficient is $\frac{(20)!}{(10)!(10)!} . 2^{10}$.

Illustration

Find the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ and of x^{-7} in $\left(ax + \frac{1}{bx^2}\right)^{11}$.
and find the relation between a and b so that these coefficients are equal.

Solution

Coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5 \frac{a^6}{b^5}$

Coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6 \frac{a^5}{b^6}$

In case these Coefficient are equal, then

$\frac{a^6}{b^5} = \frac{a^5}{b^6}$ or $a = \frac{1}{b}$ or $ab = 1$

$\therefore\ {}^{11}C_5 = {}^{11}C_6$.

Illustration

For what value of *r* the coefficients of $(r - 1)^{th}$ and $(2r + 3)^{rd}$ terms in the expansion of $(1 + x)^{15}$ are equal ?

Solution

$T_{r-1} = {}^n C_{r-2} . x^{r-2}$ $\therefore$ Coeff. is ${}^nC_{r-2}$.

$T_{2+3} = {}^n C_{2r+2} . x^{2r+2}$ $\therefore$ Coeff. is ${}^nC_{r-2}$.

Now ${}^nC_{r-2}\ {}^nC_{2r+2}$.

But if ${}^nC_p = {}^nC_q$ then $p + q = n$

$\therefore$ $(r-2) + (2r+2) = n = 15$

or $3r = 15$ $\therefore$ $r = 5$.

Illustration

If the coefficient of 4th and 13th terms in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ be equal then find the term which is independent of x .

Solution

${}^nC_3 = {}^nc_{12} \Rightarrow n = 15$

$\left(x^2 + \frac{1}{x}\right)^{15}$

we have to find the term independent of x .it will be 11th term

$T_{11} = {}^{15}C_{10}(x^2)^5 \left(\frac{1}{x}\right)^{10} = {}^{15}C_5 = 3003.$

Illustration

In the binomial expansion of $(a-b)^n$, $n \leq 5$, the sum of the 5th and 6th terms is zero . Then a/b equals.

(a) $\frac{n-5}{6}$ (b) $\frac{n-4}{5}$

(c) $\frac{5}{n-4}$ (d) $\frac{6}{n-5}$

Solution

Ans.(b)

$$T_5 + T_6 = 0$$

$$\Rightarrow\ {}^nC_4\, a^{n-4}(-b)^4 + {}^nC_5\, a^{(n-5)}(-b)^5 = 0$$

$$\therefore\quad \frac{{}^nC_5}{{}^nC_5} = -\frac{a^{n-4}}{a^{n-5}}\cdot\frac{b^4}{-b^5} = \frac{a}{b}$$

$$\text{or}\quad \frac{n-4}{5} = \frac{a}{b}$$

$$\left[\frac{\boldsymbol{{}^nC_5}}{\boldsymbol{{}^nC_4}} = \text{coeff. of } \frac{T_6}{T_5} = \frac{n-r+1}{r} = \frac{n-4}{5} \text{ for } r = 5\right]$$

Illustration

If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$ prove that its coefficient is $\frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}$.

Solution

$$T_{r+1} = {}^{2n}C_r\,(x^2)^{2n-r}.\left(\frac{1}{x}\right)^r$$

Index of x is 4n- 2r- r = p

$\therefore\quad r = \frac{4n-p}{3}$ which should be an integer.

$\therefore$ Coefficient is

$${}^{2n}C_{\frac{4n-p}{3}} = \frac{(2n)!}{[(2n+p)/3]!\left[\frac{1}{3}(4n-p)/3\right]!}\ .$$

Illustration

For $2 \leq r \geq n,\ \binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$

(c) $2\binom{n+2}{r}$ (d) $\binom{n+2}{r}$

Solution

$${}^nC_r + 2\ {}^nC_{r-1} + {}^nC_{r-2}$$

$$({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$$

$${}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r \Rightarrow (d).$$