Chapter 5 Progressions

Day – 1

Arithmetical Progression

(a):- Definition

If certain quantities increase or decrease by the same constant, then quantities from a series which is called an arithmetical progression. This constant is called common difference. For example:-

(i):- 1, 4, 7, 10,…

(ii):- 9, 6, 3, 0, - 3,…

(iii):- a, a + d, a + 2d,…

The common difference in the above three series are 3, - 3 and d respectively.

(b):- Notation

The first term of the series is denoted by a, common difference by d, the last term by l, the number of terms by n, sum of its n terms by S_n, and the nth term by T_n.

Standard Results : $T_n = a + (n – 1) d = l$.

$$S_n = \frac{n}{2}(a + l) = \frac{n}{2}[2a = (n - 1)\, d].$$

(c):- (i) Arithmetic Mean (A. M.)

The arithmetic mean between two given quantities a and b is x so that a, x, b are in A.P. i.e., x – a = b – x

or $2x = a + b \quad \therefore \quad x = \frac{a+b}{2} = A$ (Notation)

Similarly A.M. of n numbers $a_1, a_2, \ldots, a_n$ is $A = \frac{\sum a_i}{n}\ i.e., \frac{S}{n}$

(ii) *n* arithmetic means between two quantities *a* and *b*

If between two given quantities a and b we have to insert n arithmetic means $x_1, x_2, \ldots x_n$, then $a, x_1, x_2, \ldots x_n, b$ will be in A.P. In order to find the values of these means we require the common difference. The above series consists of (n + 2) terms and the last term is b and first term is a.

$\therefore \quad b = T_{n+2} = a + (n + 2 - 1)\, d$

$\therefore \quad d = \frac{b-a}{n+1}.$

$\therefore \quad x_1 = T_2 = a + d,$

$x_2 = T_3 = a + 2d, \ldots,$

$x_n = T_{n+1} = a + nd.$

On putting for d, we get

$$x_1 = a + \frac{b-a}{n+1} = \frac{na+b}{n+1} \qquad \dots (1)$$

Interchanging a and b, we get

$$x_n = \frac{nb+a}{n+1} \qquad \dots (2)$$

If we interchanging a and b in x_1 we get x_n because x_1 is first of n means from beginning and x_n is first of n means from the end.

Sum of n arithmetic means = n [single A. M.]

$$x_1 + x_2 + \cdots + x_n = \frac{n}{2}[x_1 + x_n](\text{Sum of A. P.})$$

$$= \frac{n}{2}[a + d + b - d] = n\left(\frac{a+b}{2}\right).$$

= n (A.M. of a and b)

(d):- An important note

(1):- If each of a given arithmetical progression be increased, decreased, multiplied or divided by the same non-zero quantity, then the resulting series thus obtained will also be in A.P.

Further $a_1, a_3, a_5, a_7, \dots$ or $a_2, a_4, a_6, \dots$ will also form an A.P. of common difference 2d.

Also $a_m, a_{m+n}, a_{m+2n}, a_{m+3n}, \dots$ will also form an A.P. of common difference *n*d.

(2):- Any three numbers in A.P. be taken as a – d, a, a + d. Any four numbers in A.P. be taken as a – 3d, a – d, a + d, a + 3d.

Similarly five terms in A.P. should be taken as a – 2d, a – d, a, a + d, a + 2d and six terms as a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(e):- Two important Properties of A.P

(1):- In an A.P. the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last terms

(2):- Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it :

$$a_n = \frac{1}{2}(a_{n-k} + a_{n+k}), k < n, \text{and for } k = 1,$$

$$a_n = \frac{1}{2}(a_{n-1} + a_{n+1})$$

Illustration

The fifth term of an A.P. is 1 whereas its 31st term is – 77. Find its 20th term and sum of its first fifteen terms. Also find which term of the series will be – 17 and sum of how many terms will be 20.

Solution

$T_5 = a + 4d = 1$, $T_{31} = a + 30d = -77$,

Solving the above two, we get

a = 13 and d = - 3. … (1)

$S_{15} = (n / 2)[2a + (n - 1) d]$

$= (15 / 2) [26 + 14 (-3)] = - 120.$

Let $T_n = - 17$. Then $a + (n – 1) d = - 17$.

$13 + (n - 1) (- 3) = 17$

$\therefore \quad 3n = 33$ or $n = 11$.

Let $S_n = 20$. Then $(n / 2)[2a + (n – 1)d] = 20$.

$n [2 \times 13 + (n – 1) (- 3)] = 40$, by (1).

$3n^2 – 29n + 40 = 0$

$\therefore \quad (n – 8) (3n – 5) = 0, \therefore n = 8.$

The value of *n* cannot be fractional.

Illustration

The *n*th term of a series is given to be $\frac{3+n}{4}$, find the sum of 105 terms of this series.

Solution

Putting $n = 1, 2, 3, \ldots$ in $Tn = (3 + n) / 4$

We get the series as

$$1, \frac{5}{4}, \frac{3}{2}, \ldots \therefore a = 1, d = \frac{1}{4}.$$

$$S_{105} = \frac{105}{2}\left[2.1 + (104)\frac{1}{4}\right]$$

$= (105 / 2) \times 28 = 1470.$

Illustration

Find the sum of first 24 terms of the A.P. $a_1, a_2, a_3, \ldots$ If it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

Solution

$a_5 + a_{20} = a_1 + a_{24}, a_{10} + a_{15} = a_1 + a_{24}$.

Hence the given relations reduce to

$3 (a_1 + a_{24}) = 225$, giving $a_1 + a_{24} = 75$,

Hence $S_{24} = (24 / 2) (a_1 + a_{24}) = 12 \times 75 = 900$.

Geometrical Progression

Definition

A series in which each term is same multiple of the preceding term is called a geometrical progression. In other words, a series in which the ratio of successive terms is constant is called a G.P. This constant ratio is called common ratio and is denoted by r. e.g.

(i):- 1, 4, 16, …

(ii):- 9, 6, 4, …

(iii):- $a, ar, ar^2, \ldots$

All the above series are geometrical progressions in which common ratio are $4, \frac{2}{3}$ and *r* respectively.

*n*th term of a G.P.

Let the series be $a, ar, ar^2, ar^3, \ldots$

$T_1 = a = ar^{1-1}, T_2 = ar = ar^{2-1},$

$T_3 = ar^2 = ar^{3-1}, \ldots$

$\therefore \quad T_n = ar^{n-1}.$ …(1)

Sum of *n* terms of a G.P.

Let

$S = a + ar + ar^2 + \ldots + ar^{n-1}$

$\therefore \quad r.S = [ar + ar^2 + \cdots + ar^{n-1}] + ar^n.$

Subtracting, we get

$S(1 - r) = a - ar^n$

$\therefore \quad S = \frac{a(1-r^n)}{1-r}$ ... (2)

Sum of an infinite number of terms of a G.P. when $|r| < 1$.

Since $|r| < 1$ and the number of terms is infinite,

$\therefore \lim_{n\to\infty} r^n = 0$ and hence in this case

$S_\infty = \frac{a}{1-r}$, from (2). ... (3)

Common ratio $= T_n/T_{n-1}$

i.e. divide any term by the term which precedes it.

$\therefore \quad x^2 = ab$ or $x = \sqrt{ab}$ ... (4)

n geometric means between two quantities *a* and *b*

Let $x_1, x_2, \ldots x_n$ be *n* geometric means between *a* and *b*, then $a, x_1, x_2, \ldots x_n, b$ will be a G.P. of (n + 2) terms, whose last term is *b* and term is *a*.

Single Geometric Mean Between a and b

Let x be the single geometric mean between two given quantities a and b, then a, x, b are in G.P

$\therefore \frac{x}{a} = \frac{b}{x} =$ common ration of the G.P.

$\therefore \quad b = T_{n+2} = ar^{n+1}, \; \therefore \; r = \left(\frac{b}{a}\right)^{1/(n+1)}$

$\therefore \quad x_1 = T_2 = ar,$

$x_2 = T_3 = ar^2 \ldots x_n = T_{n+1} = ar^n$

On putting the value of *r*, we shall find the *n* geometric means.

Product of *n* Geometric Means = G^n

$a, x_1, x_2, \ldots x_n, b$ is a G.P. of $n + 2$ terms.

$\therefore \quad b = T_{n+2} = ar^{n+1}$ or $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$ ... (1)

Now product of *n* means $= x_1 x_2 \ldots x_n$

$= (ar)(ar^2)(ar^3) \ldots (ar^n)$

$= a^n \, r^{1+2+3+\cdots n} = a^n \, r^{n(n+1)/2}$

$= a^n \left(\frac{b}{a}\right)^{n/2}$ by (1) $= a^{n/2} b^{n/2} = (ab)^{n/2} = \left(\sqrt{ab}\right)^n = G^n$

Where G is single G.M. of a and b.

An Important Note

1:- (a) If each term of a geometric progression be multiplied or divided by the same non-zero quantity, then the resulting series is also a G.P.

(b) In a G.P. the product of terms equidistant from the beginning and end is constant and equal to the product of the first and last terms.

2:- Odd number of terms in G.P. must be taken as

$$\ldots, ar^3, ar^2, ar, a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \ldots$$

An even number of terms in a G.P. must be taken as

$$\ldots, ar^5, ar^3, ar, \frac{a}{r}, \frac{a}{r^3}, \frac{a}{r^5}, \ldots$$

In particular three terms as $ar, a, \frac{a}{r}$ and four terms as $ar^3, ar, \frac{a}{r}, \frac{a}{r^3}$.

3:- If a_1, a_2, a_3, … and b_1, b_2, b_3, … be two G.P.'s of common ratio r_1 and r_2 respectively, then a_1b_1, a_2b_2, a_3b_3, … and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ will also form G.P. whose common ratio will be r_1r_2 and $\frac{r_1}{r_2}$ respectively.

4:- If a_1, a_2, a_3, … be a G.P. of +ive terms then log a_1, log a_2, log a_3, … will be an A.P. and conversely.

5:- Increasing and decreasing G.P.

Case I:-

Let the first term *a* be positive. Then if $r > 1$, then it is an increasing G.P. but if *r* is positive and less than 1 *i.e.* $0 < r < 1$ then it is a decreasing G.P.

Case II:-

Let the first term *a* be – ive, then $r > 1$, then it is a decreasing G.P. but if $0 < r < 1$, then it is an increasing G.P.

Illustration

$1 + (1 + x) + (1 + x + x^2) + (1 + x + x^2 + x^3) + \ldots\ldots$ to *n* terms.

Solution

*n*th term of the series is

$$T_n = 1 + x + x^2 + x^3 + \ldots n \text{ terms}$$

$$\therefore \quad T_n = \frac{1.(1-x^n)}{1-x}$$

Putting $n = 1, 2, 3, \ldots, n$ and adding, we get

$$S_n = \frac{1}{1-x}[(1+1+1+\cdots) - (x + x^2 + x^3 + \cdots)]$$

$$= \frac{1}{1-x}\left[n - \frac{x.(1-x^n)}{1-x}\right]$$

$$= \frac{1}{(1-x)^2}[n(1-x) - x(1-x^n)]$$

Illustration

If $x = 1 + a + a^2 + a^3 + \ldots$ to ∞ ($|a| < 1$) and

$y = 1 + b + b^2 + b^3 + \ldots$ to ∞ ($|b| < 1$) prove that

$$1 + ab + a^2b^2 + a^3b^3 + \cdots \text{to } \infty = \frac{xy}{x+y-1}$$

Solution

$x = \frac{1}{1-a}, y = \frac{1}{1-b}$ [summing infinite G.P.'s].

∴ $a = \frac{x-1}{x}, b = \frac{y-1}{y}$

∴ $1 + ab + a^2b^2 + \cdots \infty$

$= \frac{1}{1-ab} = \frac{1}{1-\frac{(x-1)(y-1)}{xy}} = \frac{xy}{x+y-1}$.

Illustration

The third term of a G.P. is 4. The product of first five terms is

(i) 4^3 (ii) 4^5

(iii) 4^4 (iv) None of these.

Solution

Let the G.P. be $a + ar + ar^2 + ar^3 + \ldots$

Then as given $T_3 = ar^2 = 4$. … (1)

∴ $T_1T_2T_3T_4T_5 = a.\ ar.\ ar^2.\ ar^3.\ ar^4. = a^5.\ r^{10}$

$= (ar^2)5 = 45$, by (1)

Illustration

Given x^3, y^3, z^3 are in A.P. and $\log_x y$, $\log_z x$, $\log_y z$ are in G.P. If xyz = 64, then prove that x = y = z = 4.

Solution

$2y^3 = x^3 + z^3$ or $\left(\frac{x}{y}\right)^3 + \left(\frac{z}{y}\right)^3 = 2$ ... (1)

$(\log_z x)^2 = \log_x y.\ \log_y z$

or $\left(\frac{\log x}{\log z}\right)^2 = \frac{\log y}{\log x}.\frac{\log z}{\log y} = \frac{\log z}{\log x}$

∴ $(\log x)^3 = (\log z)^3$ or $\log x = \log z$

∴ x = z

Hence from (1), $2y^3 = 2z^3$ ∴ y = z = x

But xyz = 64 ∴ $x^3 = 64$

∴ x = y = z = 4

Illustration

The sum of first two terms of an infinite G.P. is 5 and each term is three times the sum of succeeding terms. Find the series.

Solution

Here a + ar = 5

$T_p = 3(T_{p+1} + T_{p+2} + \cdots \infty)$

∴ $ar^{p-1} = 3.\frac{ar^p}{1-r}$ ∴ $1 - r = 3r$

or $r = \frac{1}{4}$ hence $a = 4$, etc.