Chapter 4 Probability

Day – 1

Some Definitions

Sample Space

The set S of all possible outcomes of an experiment (or observation) is called a sample space provided no two or more of these outcomes occur simultaneously and exactly one of the outcomes must occur whenever the experiment is performed. It should be noted that with one experiment we may succeed in associating more than one sample space. To determine a sample space, we must know precisely the aim of the experiment.

For example, consider the experiment of tossing two coins. If we are interested in whether each coin falls heads(H) or tails (T), then the possible outcomes are

$$(H,H), (H,T), (T,H), (T,T) \qquad \ldots(1)$$

On the other hand, we may be interested in whether the coins fall alike (A) or different (D). Then the possible outcomes are (A), (D).

Event

An event is a subset of a sample space. For example, for the sample space given by (1), the subset

$$A = \{(H, H), (H, T), (T, H)\}$$

Is the event that at least one head occurs. An event consisting of a single point is called a simple event.

Mutually Exclusive Events

If two or more events have an no point in common (i.e., if they cannot occur simultaneously), the event are said to be mutually exclusive.

Thus the events A and B are mutually exclusive if $A \cap B = \emptyset$

Equally Likely Events

Two events are said to be equally likely if one of them cannot be expected to occur in preference to the other.

Exhaustive Events

A set of events is said to be totally exhaustive (or simply exhaustive) if no event out side the set occurs and at least one of these events must happen as a result of an experiment.

Classical Definition of Probability

If there are n exhaustive, mutually exclusive and equally likely outcomes of an experiment and m of them are favorable to an event A, then the mathematical probability of A is defined as the ratio m /n.

Odds in Favour and Odds Against an Event

If a of the outcomes are favourable to an event A and b of the outcomes are against it as a result of an experiment. They we say that odds are a to b in favour of A, or odds are b to a against A.

Axiomatic Approach to Probability Theory

We shall be mainly concerned with discrete sample spaces, that is, with those spaces which contain only a finite number of sample points or an infinite number of points which can be arranged as a sequence a_1, a_2, a_3,…..

Axioms of Probability

Let the sample space S be the set

$$S = \{ a_1, a_2, a_3, \ldots\} = A_1 \cup A_2 \cup A_3 \cup \ldots$$

Where $A_i = \{a_i\}$ are the simple events in S.

Then to each event A in S, we assign a non-negative real number P (A), called the probability of A satisfying the following axioms:

P1: P (A) ≥0 for every events A,

P2: P (S) = 1 for the certain event S,

P3: Probabilities P (A) of any event A is the sum of the probabilities of the simple events whose union is A

Remark:- 1

(i):- If the sample space S is the union of the distinct simple events A_1, A_2, A_3,….., then it follows from axioms P_2 and P_3 that

$$P(S) = P(A_1) + P(A_2) + P(A_3) + \ldots, =1 \qquad \ldots(1)$$

(ii):- From axiom P3, we easily conclude that if A and B are mutually exclusives events, so that $A \cap B = \emptyset$

Then

$$P(A \cup B) = P(A) + P(B). \qquad \ldots(2)$$

In general, if A_1, A_2, A_3, …… are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots \qquad \ldots(3)$$

We now state theorems without proof.

Theorem:- 1

Probability of an impossible event is zero.

i.e., $P(\emptyset) = 0$

Theorem:- 2

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

Where $\bar{B}$ denoted the event complementary to event **,** that is**,** $\bar{B}$ contains the simple points of S not in B.

Theorem:- 3

$$P(\bar{A}) = 1 - P(A).$$

Theorem:- 4

(Addition theorem): Probability of A or B .

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Theorem:- 5

(i):- $P(A \cup B \cup C) = P(A) + P(B) + P(C).$

$$- P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

(ii):- $P(A_1 \cup A_2 \cup \ldots . \cup A_n)$

$$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - ..$$
$$- p(A_{n-1} \cap A_n) + P(A_1 \cap A_2 \cap A_3)$$
$$+ p(A_1 \cap A_2 \cap A_4) + \ldots$$
$$+ P(A_{n-2} \cap A_{n-1} \cap A_n)$$
$$+ . + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n).$$

Remark:- 2

Since probability of an event is a non – negative numbers, it follows from theorem 4 that

$$P(A \cup B) \leq P(A) + P(B) \quad \ldots (4)$$

The inequality (4) holds in general. Thus for arbitrary events A_1, A_2, A_3,….., we have

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots ..) \leq P(A_1) + P(A_2) + P(A_3) + \ldots \quad \ldots (5)$$

The inequality (5) is known as Boole's inequality.

Conditional Probability

A word of explanation is necessary to understand conditional probability. Suppose a pair of dice is thrown. Here the sample space S consists of 36 points (1,1), (1,2),……(1,6); (2,2), …(2,6);…; (6,1)(6,2),(6,6). Suppose that we now ask the question, "If a of dice shows as an even sum, what is the probabilities that this sum is less then 6?" Here we are restricting the sample space to a subset of points corresponding to even sum only (18 such points) and asking, " which of these possible points (outcomes) represents a sum less than 6?" There are 4 such points viz., (1,1), (1,3), (3,1), (2,2). Since all these outcomes are equally likely, the required probability $P = \frac{4}{18} = \frac{2}{9}$

Observe that here we have imposed the condition that the sum, x + y, is even (event A), and then asked the probability for x + y to be less than 6 (event B). We denote this conditional probability by the symbol P (B/A), Which is read " the probability of the event B under the condition that the event A has already happened. A little consideration will show that the 2/9 was obtained when we divided the number n (A ∩ B) of points in the subset

A ∩ B = {(x, y): x + y is even and < 6}

By the number, n (A), of points in the subset

A = {(x, y): x + y is even}, so that

$$P\left(\frac{B}{A}\right) = \frac{n(A \cap B)}{n\ (A)} = \frac{n\frac{A \cap B}{n(S)}}{\frac{n(A)}{n(S)}} = \frac{P(A \cap B)}{P(A)} \quad \ldots (1)$$

Where n (S) is the number of points in the entire sample space. Note that here it is assumed that

P(A) ≠ 0

Although the equation (1) has been obtained for the special case where all points of S have the same probability 1/36, it will be found to hold in those cases as well where the equality of probability does not hold.

Thus equation (1) constitutes our formal definition of conditional probability of B on the hypothesis A (or for given A).

Multiplication Theorem

P(A ∩ B) = P (A) P (B/A), i.e., the probability of the simultaneous occurrence of the evens A and B equals the probability of A multiplied by the conditional probability of B that A has already occurred.

Independent events

Of the happening of the event B is not influenced or conditioned by a second event A (P(A) ≠ 0) so that

$$P(B/A\ ) \ = P(B)$$

The B is said to be independent of A.

Theorem:- 1

If P(A) ≠ 0 and P(B) = 0 and B is independent of A, then A is independent of B. In this case, we say that A and B are mutually independent.

Theorem:- 2

Two events A and B are mutually independent if and only if

$$\mathrm{P}(A\ \cap B) = \ P(A)\, P(B)$$

Provided P(A) ≠ 0 and P (B) ≠ 0.

The generalization of this theorem is as follows: The events A_1, A_2, ….An are mutually independent (or simply independent) if and only if the multiplication rule

$$P(A_1 \cap A_2\ \cap \ldots .\cap A_k) = P(A_1)P(A_2) \ldots P(A_k) \quad \ldots (3)$$

Holds for every k – tuples of events; k = 2, 3, …, n.

If (3) holds for k = 2 and may or may not hold for k = 3, 4, …,n, then events A_1, A_2,…, A_n are said to be pair wise independent. Thus mutually independent events are pair wise independent but not conversely.

Theorem:- 3

If the event A and B are mutually independent then the events A and $\bar{B}$are mutually independent.

Theorem:- 4

If A and B are mutually independent event such that P (A) ≠ 0 and P (B) ≠ 0, then the events A and B have at least one common simple point.

Important Remarks

(a):- Let E_1 and E_2 be two independent events with respective probabilities p_1 and p_2, then the probability p of the event E_1 to occur and that of E_2, not to occur is given by p = p_1 (1 – p_2).

(b):- If there are n mutually independent events E_1, E_2, E_3, ….En with respective probabilities p_1, p_2, …, p_n, then the probability of m events E_1, E_2, …E_m to occur and that of the remaining (n – m) events E_{m+1}.

(c):- The probability that none of the n events occur is given by (1 – p_1) (1 – p_2) … (1 – p_n).

(d):- The probability that at least one of the n events will occur is given by 1 – (1 – p_1) (1 – p_2)…..(1 – p_n).

Partition of a sample Space

Let A_1, A_2, …, A_n be subsets of a sample space S. Then these subsets are said to form a partition of S if the following conditions hold:

(i):- Each A_i is a proper subset of S, that is

$$A_i \subset S, i = 1,2, \ldots . n \text{ and } A_i \neq S.$$

(ii):- $A_1 \cup A_2 \cup \ldots . \cup A_n = S.$

(iii):- The subsets A_i are pair wise disjoint, that is,

Theorem of Total Probability for Compound Events

Theorem:- 1

Let {A1, A2, …, An} be any partition of a sample space S and let A be an event. Then

$$P(A) = \sum_{j=i}^{n} P(A_i) P\left(\frac{A}{A_i}\right) \qquad \ldots (1)$$

Provided P(A_i) ≠ 0, i = 1, 2, …, n.

Theorem:- 2 (Baye's Rule)

Let the set of events $\{A_1, A_2, ..., A_n\}$ from a partition of the sample space S, where $P(A_i) \neq 0$, i =1,2,…n. Then for any event A for which $P(A) \neq 0$ and for $1 \leq k \leq n$.

$$P(A_k/A) = \frac{P(A`_k)P(A/A_k)}{\sum_{i=1}^{n} P(A_i)P(A/A_i)}$$

Baye's theorem can me put in the following useful form.

Let $S = A_1 \cup A_2 \cup ....\cup A_n$ where A_1, are simple events. Then clearly A_i form a partition of S. If A is any non – empty subset of S, then for each integer k ($1 \leq k \leq n$).

$$P(A_k\ A) - \frac{P(A/A_k)P(A_k)}{\sum_{i=1}^{n} P(A/A_i)P(A_i)} \qquad ... (2)$$

Binomial Distributions

Theorem

If X is a random binomial variety, then X ~ B (n,p), n, p being the parameters of the binomial probability distribution.

Then expected value of X i.e., E(X) = np

And variance of X i.e., var (X) = npq

Use of Multinomial Theorem

Suppose a die has m and marked with the number 1, 2, 3, …,m and such n dices are thrown.

Then the probability that the sum of the numbers shown on the upper faces is equal to s is given by

The coefficient of x^s in the expansion of

$$\frac{(x+x^2+x^3+ ......+x^m)^n}{m^n}$$

Alternative Concept of Inverse Probability (Baye's rule)

Sometimes in probability Theory, we come across quite often, the problems of the following nature. Suppose when an event has happened, it may be due to some one of the n causes. The apriority probability of these causes is estimated as $P_1, P_2, ..., P_n$.

Let pr denoted the probability of the happening of the event due to the r th cause, then the antecedent probability that the event follows from rth cause is pr pr.

Let Qr denote the a posteriori probability that the rth cause was the true one, then Qr is proportional to the probability given by

$$\frac{Q_1}{p_1P_1} = \frac{Q_2}{p_2P_2} .... = \frac{Q_r}{p_rP_r} = \cdots = \frac{Q_n}{p_nP_n}$$

$$= \frac{\sum_{r=1}^{n} Qr}{\sum_{r=1}^{n} p_rP_r} = \frac{1}{\sum_{r=1}^{n} p_rP_r}$$

Thus $$Q_r = \frac{p_rP_r}{\sum_{r=1}^{n} p_rP_r}$$

Illustration

Five persons entered the lift cabin on the ground floor of 8 – floor house. Suppose that each of them independently and with equal probability, can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floors.

Solution

Besides the ground floor, there are seven floors. The total number of ways in which each of the five persons can leave cabin at any the seventh floors = 7^5.

And the favourable number of ways, that is, the number of ways in which the 5 persons leave at different floors is $^7P_5$

∴ The required probability $= {}^7P_5/7^5$

Illustration

A student is given a true – false exam. with 10 questions. If he gets 8 or more correct answers he passes the exam. Given that he guesses at the answer to each question, compute the probability that he passes the exam.

Solution

n = total no. of ways = 2s = 1024.

Since each answer can be true or false.

And m = favourable numbers of ways

$$^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10} = 45 + 10 + 1 = 56.$$

Since to pass the exam, he must gives 8 or 9 or 10 true answers.

Illustration

There are 5 pairs of shoes in a shoe rack. Four shoes are drawn one by one at random. Find the probability that at least one pair of shoes is drawn.

Solution

The probability that at least one pair is drawn = – 1 (None of them are pair)

$$= 1 - \frac{10}{10}.\frac{8}{9}.\frac{6}{8}.\frac{4}{7} = 1 - \frac{8}{21} = \frac{13}{21}$$

Illustration

There are four machines and it is known that exactly two of them are faulty. They are tested, one by one, in a random order till both the faulty machines are identified. Then the probability that only two tests are needed is

(a) 1/3 (b) 1/6

(c) ½ (d) 1/4

Solution

(b)

A = event that two defective machines are identified in first two tests out of four machines.

∴ $$P(A) = \frac{^2C_2}{^4C_2} = \frac{1}{6}$$