Chapter 3 Permutations and Combination

Day – 1

Definitions of Permutation

Each of the different arrangements which can be made by taking some or all of a number of things is called a **permutations.**

Definitions of Combination

Each of the different groups or selections which can be made by taking some or all of a number of things (irrespective or order) is called a **combination.**

Fundamental Theorem

If there are m ways of doing a thing and for each of the m ways there are associated n ways of doing a second thing then the total number of ways of doing the two things will be mn.

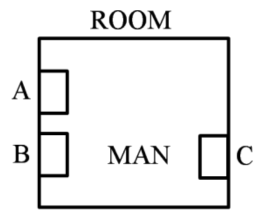
As and example suppose six subjects are to be taught in four periods. For the first period we can put any of the six subjects i.e., there are 6 ways of filling the first period. For the second period we are left with remaining subjects and hence there are 5 ways of filling the second period. Hence the number of ways in which first two periods can be filled up is $6 \times 5 = 30$ ways.

Illustration

Suppose there are 3 doors in a room, 2 on one side and 1 on other side. A man want to go out from the room.

Solution

Obviously he has '3' options for it. He can come out by door 'A' or door 'B' or door 'C'.

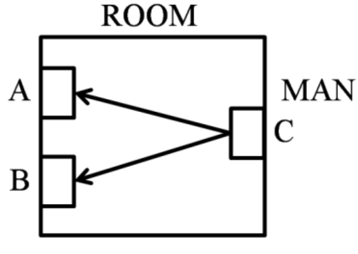


Illustration

Suppose a man wants to cross-out a room, which has 2 doors on one side and 1 door on other site.

Solution

He has 2 x 1 = 2 ways for it.



Important Results

(a):- Number of permutations of n dissimilar things taken r at a time.

$$^{n}P_{r} = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)$$

The last factor is [n – (r – 1)] = n – r + 1.

Where n! = 1.2.3……n.

Note that n! = n.(n – 1)!

= n(n – 1).(n – 2)!etc.

Note:- In the above repetition was not allowed so that if we could fill the first place in n ways then second could be filled in (n – 1), third in (n – 2)….are rth in (n – r + 1) ways. Hence the total by fundamental theorem was

$n(n-1)(n-2)\ldots.(n-r+1)$ ways

Now if repetition is allowed.

In this case each of the r places can be filled in n ways. Hence by fundamental theorem all the r places can be filled in n.n.n.n….r times = n^r ways.

(b):- Number of permutations of n dissimilar things taken all at a time.

$$^{n}P_{n} = n(n-1)(n-2)\ldots\ldots\ldots[n-(n-1)]$$

= n(n – 1)(n – 2)…….3.2.1. = n!

(c):- Number of combinations of n dissimilar things taken r at a time.

$$^{n}C_{r} = \frac{n!}{(n-r)!r!} = \frac{^{n}P_{r}}{r!} \quad \text{or} \quad r!\ ^{n}C_{r} = {}^{n}P_{r}$$

(d):- Number of combinations of n dissimilar things taken all at a time.

$$^{n}C_{n} = \frac{n!}{n!(n-n)!} = \frac{1}{0!} = 1 \qquad \because 0! = 1$$

(e):- If out of n things p are exactly alike of one kind, q exactly alike of second kind and r exactly alike of third kind and rest all different, then the number of permutations of n things taken all at a true.

$$= \frac{p!}{p!.q!.r!}$$

(f):- Number of circular permutations of n different things taken all at a time.

Here having fixed one thing the remaining (n – 1) things can be arranged round the table in (n – 1)! Ways.

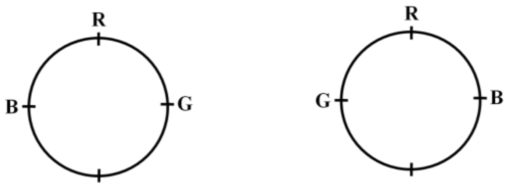
Note:- If however n persons are to be arranged in a row then as show in (b) the number of arrangements was nth whereas for a circular table as shown above the number of arrangements is (n – 1)!.

Particular Case: Necklace:- Number or arrangements of n beads all different to form a necklace or on a circular wire will be ½(n – 1)! As explained in the figure below:

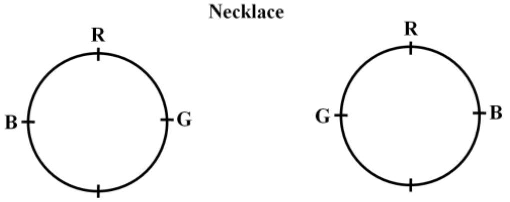
R = Ram. G = Ganesh. B = Brahma
R = Red. G = Green. B = Blue

The above seating arrangements of these person on the round table are different as shown in upper figure and that is why we say (n – 1)! Clockwise and anticlockwise make different arrangements.

Round Table



Necklace



The above two arrangements of three flowers to form a necklace is the same because on the necklace we get the same arrangements and that is why we say the total number of arrangements of n beads for forming a necklace is 1/2(n – 1)!. Here clockwise or anticlockwise does not change the character of the necklace. It remains the same.

(g):- If some or all of n things be taken at a time then the number of combinations will be

$$2^n - 1. \because {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_r = 2^n - 1.$$

(h):- ${}^nC_r = {}^nC_{n-r}$

(i):- ${}^nC_{r1} = {}^nC_{r2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$.

(j):- ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

(k):- Number of combinations of n dissimilar things taken r at a time when p particular things always occure.

$$= {}^{n-p}C_{r-p}$$

(l):- Number of combinations of n dissimilar things taken r at a time when p particular things never occur.

$$= {}^{n-p}C_r$$

(m):- Number of permutations of n dissimilar things taken r at a time when p particular things always occure.

$$= {}^{n-p}C_{r-p}.r!$$

(n):- Number of permutations of n dissimilar things taken r at a time when p particular things never occure.
$= {}^{n-p}C_r\, r!$.

(o):- Division Into Groups

(i):- The number of ways in which m + n things can be divided into two groups containing m and n things respectively$= \frac{(m+n)!}{m!n!}$.

(ii):- If n = m, the groups are equal and in this case the number of different ways of subdivision $= \frac{2m!}{m!m!2!}$. For in any one way it is possible to interchange the two groups without obtaining new division.

(iii):- But if 2m things are to be divided equally between two person then the number of divisions $= \frac{2m!}{m!m!}$.

(iv):- Similarly the number of divisions of m + n + p things into groups of m, n and p things respectively $= \frac{(m+n+p)!}{m!m!n!p!}$.

(v):- If 3m things are divided into three equal groups, then the number of divisions $= \frac{(3m)!}{m!m!m!3!}$.

(vi):- But if 3m things are to be divided amongs three persons, then the number of divisions $= \frac{(3m)!}{m!m!m!}$.

(p):- Number of permutations of n dissimilar things taken r at a time when each things an be repeated once, twice,….upto r times = n^r.

Note:- from above we conclude the following: While distributing certain m things equally amongst four persons, the things which has gone to one person shall not go to the other and hence in this case we shall not divide by 4!. But if however we from four equal groups then we shall divide by4!.

For example:-

If 20 different things are to be equally distributed amongst 4 persons the answer will be $\frac{(20)!}{(5!)^4}$ but if 20 things are to form 4 equal groups then the answer will be $\frac{(20)!}{(5!)^4\, 4!}$. This 4! Corresponds to the number of these four groups.

Illustration

If ${}^{n}C_{10} = {}^{n}C_{15}$, find ${}^{27}C_n$.

Solution

${}^{n}C_{10} = {}^{n}C_{15}$.

We know ${}^{n}C_r = {}^{n}C_{n-r}$

$\therefore$ ${}^{n}C_{10} = {}^{n}C_{n-10}\, {}^{n}C_{15}$, by (1)

n – 10 = 15 or n = 25

or If ${}^{n}C_x = {}^{n}C_y$ then $x + y = n$.

Hence from (1), we get 10+15 = 25 = n

$\therefore$ ${}^{27}C_n = {}^{27}C_{25} = {}^{27}C_{27-25} = {}^{27}C_2$

$$\frac{27\times 26}{1\times 2}=27\times 13=1351$$

Illustration

If ${}^{n}C_6 : {}^{n-3}C_3 = 33:4$, find n.

Solution

$$\frac{{}^{n}C_6}{{}^{n-3}C_3}=\frac{33}{4}$$

or $\frac{n!}{6!(n-6)!}.\frac{3!(n-6)!}{(n-3)!}=\frac{33}{4}$

or $\frac{n(n-1)\,(n-2)}{6.5.4}=\frac{33}{4}$

or $n(n-1)(n-2)=30\times 33=11\times 3\times 3\times 10$

$=11\times 10\times 9=11(11-1)(11-2)$

Hence clearly n = 11

Illustration

(i) How many words can be formed by taking 4 letters at a time out of the letters of the word MATHEMATICS?

Solution

We can choose 4 letters from the 11 listed in part (a) as under.

(i):- All the four different:-

We have 8 different types of letters and out of theses 4 can be arranged in

$${}^{8}P_4=\frac{8!}{4!}=8\times 7\times 6\times 5=1680$$

or 4 can be selected in

${}^{8}C_4$ ways and arranged in

$${}^{8}C_4\times 4!=\frac{8!}{4!4!}.4!=\frac{8!}{4!}=1680 \qquad \text{... (1)}$$

(ii):- Two different and two alike:-

We have 3 pairs of like letters out of which one pair can be chosen in ${}^{3}C_1=3$ ways. Now we have to choose two out of the remaining 7 different types of letters which can be done in

$$= {}^{7}C_2=\frac{7!}{5!2!}=\frac{7\times 6}{2}=21 \text{ ways.}$$

Hence the total number of groups, of 4 letters in which 2 are different and 2 are alike is 3 × 21 = 63 groups.

Let one such group be M,H,M,I.

Each such group has 4 letters out of which 2 are alike and they can be arranged amongst themselves in $\frac{4!}{2!}=12$ ways. Hence the total number of words is

$$63\times 12=756. \qquad \text{... (2)}$$

(iii):- Two alike of one kind and two alike of other kind:-

Out of 3 pairs of like letters we can choose 2 pairs in

${}^{3}C_2$ ways $=3$ ways.

One such group is *MM AA*.

These four letters out of which 2 are alike of one kind and 2 alike of other kind, can be arranged in $\frac{4!}{2!2!} = 6$ ways.

Hence the total number of words of this type is

$3 \times 6 = 18$ ... (3)

Therefore from (i), (ii) and (iii) the number of 4 letter words is

1680 + 756 + 18 = 2454.

by (1), (2) and (3)

Illustration

Number plates of cars must contain 3 of the alphabet denoting the place and area to which its owner belongs. This is to be followed by a three-digit number. How many different number plates can be formed if :

(i) repetition of letters and digits is not allowed.

(ii) repetition of letters and digits is allowed.

Solution

There are 26 letters of alphabet and 10 digits from 0 to 9.

Repetition not allowed:-

3 letters 26 × 25 × 24

3 digit numbers ${}^{10}P_3 - {}^{9}P_2$ (beginning with zero)

= 10 . 9 . 8 – 9 . 8 = (10 – 1) = 9 . 9 . 8

∴ $\text{Number of plates} = 26 \times 25 \times 24 \times 9 \times 9 \times 8$

= 10108800.

Repetition allowed:-

3 letters 26 × 26 × 26

3 digit numbers = 9 × 10 × 10 = 900

1^{st} place can't be filled by 0, 2^{nd} can be filled by 10 and so is 3^{rd} as repetition is allowed.

∴ $\text{Number of plates} = 26 \times 26 \times 26 \times 900$

= 158184400.