Chapter 2 Theory of Quadratic

Day – 1

Roots of the Equation

$ax^2 + bx + c = 0.$

Multiplying both sides by 4a, we get

$4a^2x^2 + 4abx = -4ac$

Add b^2 to both sides

$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$

or $(2ax + b)^2 = b^2 - 4ac$

Take square root

$2ax + b = \pm\sqrt{b^2 - 4ac}.$

∴ $\boldsymbol{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}.$

Sum and Product of the Roots

If α and β be the roots, then

$$\alpha = \frac{-b\pm\sqrt{b^2-4ac}}{2a}, \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Sum of the roots $= \alpha + \beta = -\frac{2b}{2a} = -\frac{b}{a}$

or $\alpha + \beta = -\frac{b}{a} = -\frac{\text{coeff .of x}}{\text{coeff .of x}^2}$

Product of roots

$$= \alpha\beta = \frac{(-b)^2 - \left[\sqrt{b^2-4ac}\right]^2}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

or $\alpha\beta = \frac{\textbf{constant term}}{\textbf{coeff.of x}^2}.$

To Find The Equation Whose Roots are α and β

The required equation will be

$(x - \alpha)(x - \beta) = 0$

or $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or $x^2 - Sx + P = 0$

Where S is sum and P is product of roots. If the roots be 3, - 7

Then S = Sum = 3 – 7 = - 4,

P = Product = - 21

∴ Equation is $x^2 - Sx + P = 0$

or $x^2 - (-4)x + (-21) = 0$

or $x^2 + 4x - 21 = 0.$

Nature of the Roots

The roots of the equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

The expression $b^2 - 4ac$ is called discriminant.

(a):- If $b^2 - 4ac \geq 0$, roots are real.

(i):- If $b^2 - 4ac > 0$, then the roots are real and unequal.

(ii):- If $b^2 - 4ac = 0$, then the roots of the equation are real and equal.

In this case, each root $= \frac{-b \pm 0}{2a} = -\frac{b}{2a}$.

(iii):- Also if $b^2 - 4ac$ be a perfect square then the roots are rational and in case it be not a perfect square then the roots are irrational.

(b):- If $b^2 - 4ac < 0$ *i.e.* – ive, then $\sqrt{b^2 - 4ac}$ is imaginary.

Therefore the roots are imaginary and unequal. Imaginary or irrational roots of the equation $ax^2 + bx + c = 0$ where *a, b, c* are all real numbers, will occur in conjugate pairs *i.e.*, if $2 + 3i$ is a root then $2 - 3i$ will also be a root. But if however the coefficients *a, b, c* are not all real or any of these is non-real or irrational, then it is not necessary that the roots will occur in conjugate pairs.

Particular Case. If $a + b + c = 0$, then $x = 1$ is a root of the equation $ax^2 + bx + c = 0$ and if $a - b + c = 0$, then $x = -1$ is a root of above.

Identity. In case any quadratic equation $ax^2 + bx + c = 0$ has more than two roots then it will be an identity which implies that all the coefficients *a, b, c* are zero. In this case, the equation is satisfied by all the values of *x*.

(a) Symmetric Functions of the Roots

If α and β are the roots of $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \qquad \text{... (1)}$$

$$\alpha\beta = \frac{c}{a}. \qquad \text{... (2)}$$

$$\therefore \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\beta = \frac{b^2 - 2ac}{a^2}$$

$$\alpha - \beta = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \frac{\sqrt{b^2 - 4ac}}{a}$$

$$\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = -\frac{b}{a}.\frac{\sqrt{b^2 - 4ac}}{a}$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$= -\frac{b^3}{a^3} + \frac{3bc}{a^2} = -\frac{b(b^2 - 3ac)}{a^3}$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)^2 - \frac{2c^2}{a^2}$$

(b) Transformation of Equations

Let α, β be the roots of the equation

$ax^2 + bx + c = 0.$ … (1)

To find the equation whose roots are

(i) Negative of the Roots of (1)

The required roots are - α, - β.

This is effected by putting y = - α = - x.

or x = - y in (1)

∴ $ay^2 - by + c = 0$

or $ax^2 - bx + c = 0$ … (2)

(ii) Reciprocal of the Roots of (1)

The required roots are $\frac{1}{\alpha}, \frac{1}{\beta}$.

This is effected by putting $y = \frac{1}{\alpha} = \frac{1}{x}$

or $x = \frac{1}{y}$ in (1)

$\frac{a^2}{y} + \frac{b}{y} + c = 0$

or $cy^2 + by + a = 0$ … (3)

(iii) Square of the Roots of (1)

The required roots are α^2, β^2.

This is effected by putting $y = \alpha^2 = x^2$

or $x = \sqrt{y}$ in (1)

$ay + c = -b\sqrt{y}$. Square both sides

$a^2y^2 + c^2 + 2acy = b^2y$

or $a^2y^2 + (2ac - b^2)y + c^2 = 0$

(iv) Cube of the Roots of (1)

The required roots are α^3, β^3.

This is effected by putting $y = \alpha^3 = x^3$

or $x = y^{1/3}$ in (1)

∴ $ay^{2/3} + by^{1/3} = -c$. Cube both sides

$a^3y^2 + b^3y + 3aby^{2/3}\, y^{1/3}\left(ay^{\frac{2}{3}} + by^{\frac{1}{3}}\right) = -c^3$

or $a^3y^2 + b^3y + 3aby\,(-c) = -c^3$

or $a^3y^2 + y(b^3 - 3abc) + c^3 = 0$

(v) Increased by *i.e.*, α + *h*, β + *h*

This is effected by putting $y = \alpha + h = x + h$ or by putting $x = y - h$ in the given equation (1)

∴ $a(y-h)^2 + b\,(y-h) + c = 0$

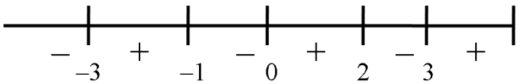
$ay^2 + y(b - 2ah) + (ah^2 - bh + c) = 0$

Sign of the Expression

Lets take this example

(x + 3) (x + 1) x (x - 2) (x - 3) …

Put the given expression equal to zero and find the values of x and write them in ascending order of magnitude i.e. – 3, - 1, 0, 2 and 3. Now on the real line as shown below:



Rule:- Start with extetrme right with + ive sign and move towards left and write opposite signs alternately.

x > 3, + ive; 2 < x < 3, - ive; 0 < x < 2, + ive;

- 1 < x < 0, - ive; - 3 < x < - 1, + ive; x < - 3, - ive

Modulus Function

|x| = x if x is + ive and |x| = - x if x is – ive

∴ $|x - a| = x - a$ if $x - a \geq 0$ or $x \geq a$

$= -(x - a)$ if $x - a < 0$ or $x < a$

Greatest Integer Function

$[x] = n$ if $n \leq x < n + 1$

Common Roots

If α is a common roots of the quadratics

$f_1\,(x) = a_1x^2 + b_1x + c_1 = 0$

$f_2\,(x) = a_2x^2 + b_2x + c_2 = 0,$

Then α will satisfy both and by the method of cross multiplication,

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1} = \text{k (say)}$$

I II III

The common root is given by

$$\alpha = \frac{\text{I}}{\text{II}} = \frac{\text{II}}{\text{III}}$$

The required condition is given by

$(\text{II})^2 = (\text{I})\,(\text{III})$

or $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$

Both the roots be common

In this case both sum and product will be same

$\because \quad -\frac{b_1}{a_1} = -\frac{b_2}{a_2}$ and $\frac{c_1}{a_1} = \frac{c_2}{a_2}$

or $\quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Inequalities

(a):- $x^2 > a^2$ or $(x^2 - a^2) = +\text{ive}$

or $[x - (-a)]\,(x - a)$ + ive

If $x < -a$ or $x > a$

(b):- $x^2 < a^2$ if $-a < x < a$

(c):- $a^2 < x^2 < b^2$

$\therefore \quad a < x < b$ or $-b < x < -a$

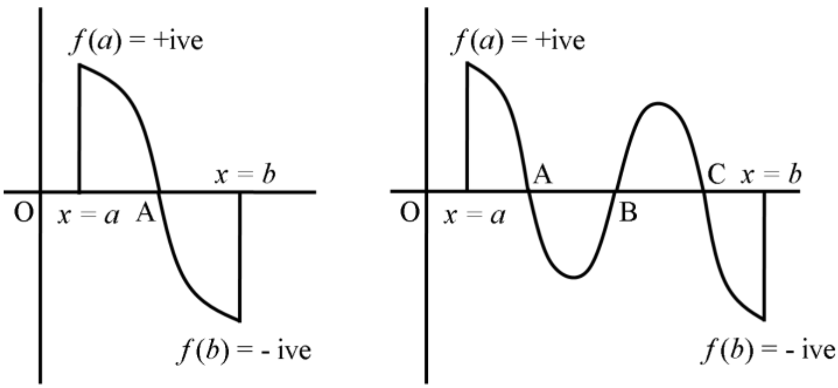
Certain Important Definitions

1. Identity:- A quardratic equation $ax^2 + bx + c = 0$ is satisfied by only two values of x. However if it is satisfied by more than two values of x, then it is called an identity and in this case it is satisfied by every value of x in the domain of x. Also in this case $a = 0$, $b = 0$, $c = 0$. Consider the equation

$$(x - 2)^2 - (x^2 - 4x + 4) = 0.$$

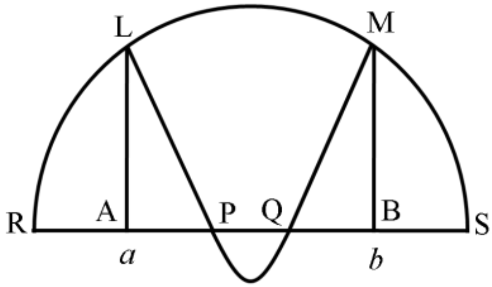
It is satisfied by all values of x and it reduces to the form $0x^2 + 0x + 0 = 0$ i.e. $a = 0$, $b = 0$, $c = 0$.

2. If $f(a)$ and $f(b)$ are of opposite signs then at least one or in general odd number of roots of the equation $f(x) = 0$ lie between a and b.



As is clear from the figure, in either case there is a point P or Q at x = c where tangent is parallel to x-axis $i.e. f'(x) = 0$ at x = c.

4. If $f(a)$ and $f(b)$ are of the same sign then either no root or an even number of roots will exist which either lie between a and b or lie outside a and b. Both at P and Q, $y = f(x) = 0$ they lie between a and b. But at R and S, $y = f(x) = 0$ and they lie outside a and b as shown :



5. Imaginary and irrational roots occur in conjugate pairs i.e. if – 3 + 2i is a root then – 3 – 2i will be the second. Similarly if $5-2\sqrt{7}$ is a root then $5+2\sqrt{7}$ will also be a root.

6. If $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots = 0$

$$= a_0(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$$
$$= a_0(x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \cdots)$$

Then $S_1 = -\frac{a_1}{a_0}$, $S_2 = \frac{a_2}{a_0}$,

$S_3 = -\frac{-a_3}{a_0}, \ldots S_r = (-1)^r\frac{a_r}{a_0}$,

Where $S_1 = \sum\alpha, S_2 = \sum\alpha_1\ \alpha_2,\ S_3 = \sum\alpha_1\ \alpha_2\ \alpha_3$ etc.

If the roots of the equation

$ax^3 + bx^2 + cx + d = 0$ be α, β, γ then

$\sum\alpha = -\frac{b}{a}, \sum\alpha\beta = \frac{c}{a},\ \ \alpha\,\beta\,\gamma = -\frac{d}{a}$ or if the roots of

$ax^4 + bx^3 + cx^2 + dx + e = 0$ be $\alpha, \beta, \gamma, \delta$ then

$\sum\alpha = -\frac{b}{a},\ \ \sum\alpha\,\beta = -\frac{c}{a}$,

$\sum\alpha\,\beta\,\gamma = -\frac{d}{a},\ \ \alpha\,\beta\,\gamma\,\delta = \frac{e}{a}$.

7. Lagrange's Identity

$$(a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2$$
$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}^2 + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}^2 + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}^2$$

8. In an inequality you can always multiply or divide by a + ive quantity but not by a – ive quantity.

Multiplying by a – ive quantity or taking reciprocal will reverse the inequality.

$e.g., a > b \Rightarrow -a < -b$ or $\frac{1}{a} < \frac{1}{b}$

9. $a^2 + b^2 + c^2 - ab - bc - ca$

$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = +\text{ive always}$$

It will be zero only when $a - b = 0, b - c = 0, c - a = 0$ *i.e.* $a = b = c$.

10. Ratio Proportion

We have already stated elsewhere in this book that if $\frac{a}{b} = \frac{c}{d}$ then each is equal to

$$\frac{a+c}{b+d} \quad \text{or} \quad \left(\frac{a^2+c^2}{b^2+d^2}\right)^{\frac{1}{2}} \quad \text{or} \quad \left(\frac{a^3+c^3}{b^3+d^3}\right)^{\frac{1}{3}}$$

$$\text{or} \quad \left(\frac{ac}{bd}\right)^{\frac{1}{2}} \quad \text{or} \quad \frac{pa^n+qc^n}{pb^n+qd^n} \quad \text{and so on.}$$

We can also have minus sign in place of plus and we can have as many ratios as we like instead of only two.

11. Cubic Equation

$$ax^3 + bx^2 + cx + d = 0$$

If it roots be α, β, γ then

$$S_1 = \sum \alpha = -\frac{b}{a},\ S_2 = \sum \alpha\beta = \frac{c}{a},\ S_3 = \alpha\beta\gamma = -\frac{d}{a}$$

12. **Biquadratic Equation**

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$S_1 = \sum \alpha = -\frac{b}{a}$$

$$S_2 = \sum \alpha\beta = (\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = \frac{c}{a}$$

$$S_3 = \sum \alpha\beta\gamma = \alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = -\frac{d}{a}$$

$$S_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

Illustration

If α and β are the roots of $ax^2 + bx + c = 0$, find the values of following :

(a) $\frac{1}{a\alpha+b} + \frac{1}{a\beta+b}$ (b) $\frac{\beta}{a\alpha+b} + \frac{\alpha}{a\beta+b}$

Solution

Since α and β are the roots of

$$ax^2 + bx + c = 0$$

$$a\alpha^2 + b\alpha + c = 0 \quad \text{or} \quad a\alpha + b = -\frac{c}{\alpha}.$$

$$a\beta^2 + b\beta + c = 0 \quad \text{or} \quad a\beta + b = -\frac{c}{\beta}.$$

Also $\alpha + \beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}.$

(a) $\frac{1}{a\alpha+b} + \frac{1}{a\beta+a} = -\frac{\alpha}{c} - \frac{\beta}{c}$

$$= -\frac{1}{c}(\alpha+\beta) = -\frac{1}{c}\left(-\frac{b}{a}\right) = \frac{b}{ac}.$$

(b) $\frac{\beta}{a\alpha+b} + \frac{\alpha}{a\beta+b} = -\frac{\alpha\beta}{c} - \frac{\alpha\beta}{c} = -\frac{2}{c}.\frac{c}{a} = -\frac{2}{a}.$

Illustration

If α and β are the roots of $x^2 - p(x + 1) - c = 0$, show that $(\alpha + 1)(\beta + 1) = 1 - c$. Hence prove that $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = 1.$

Solution

The given equation is $x^2 - px - (p + c) = 0$

$\therefore \quad \alpha + \beta = p,\ \alpha\beta = -(p + c)$

$(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1$

$= - p - c + p + 1 = 1 - c.$ …(1)

Again $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$

$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$

$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)}$, by (1)

$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} = \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1.$

Illustration

In a triangle $PQR, \angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$, $(a \neq 0)$, then

(a) $a + b = c$ (b) $b + c = a$

(c) $a + c = b$ (d) $b = c$

Solution

$t_1 + t_2 = -\frac{b}{a}, t_1 t_2 = \frac{c}{a}$

where $t_1 = \tan\frac{P}{2}, t_2 = \tan\frac{Q}{2}$

$P + Q = \pi - R = \frac{\pi}{2}$ as $R = \frac{\pi}{2}$

$\therefore \quad \frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$

$\frac{t_1+t_2}{1-t_1 t_2} = \frac{\pi}{4} = 1$

or $\quad -\frac{b}{a} = 1 - \frac{c}{a} \Rightarrow a + b = c$