# Chapter 1 Complex Number

## Day – 1

### Complex Numbers

A complex numbers may be defined as an ordered pair of real numbers and may be denoted by the symbol (x, y). If we write z = (x, y), then x is called the real part and y the imaginary part of the complex number z and may be denoted by Re(z) and Im(z) respectively.

It is clear from the definition that two complex numbers (x, y) and (x', y') are equal if and only if x = x' and y = y'. We shall denote the set of all complex numbers by the letter C.

### Sum of two complex numbers

The sum of two complex numbers (x, y) and (x', y') is defined by the equality

$$(x, y) + (x', y') = (x + x', y + y')$$

### Product of two complex numbers

The product is defined by the equality

$$(x, y)\,(x', y') = (xx' - yy', xy' + yx')$$

**The symbol i(iota):**

- *i* is the square root of the real number – 1.
- It is customary to denote the complex number (0, 1) by the symbol i.

With this notation

$$i^2 = (0, 1)\,(0, 1)$$
$$= (0.0 - 1.1, 0.1 + 1.0) = (-1, 0)$$

So that *i* may be regarded as the square root of the real number – 1.

Using the symbol *I*, we may writ the complex number (x, y) as x + *i*y. For, we have

$$x + iy = (x, 0) + (0, 1)\,(y, 0)$$
$$= (x, 0) + (0.\,Y - 1.0, 0.0 + 1.y)$$
$$= (x, 0) + (0, y) = (x + 0, 0 + y) = (x, y).$$

**Remark:-**

$\because \quad i = \sqrt{-1}$ and $i^2 = -1,$ we have

$$\left(\sqrt{-1}\right)^2 = \sqrt{-1}.\sqrt{-1} = -1 \qquad \text{... (i)}$$

Again since

$$\left(\sqrt{a}.\sqrt{-1}\right)^2 = \left[\sqrt{a}.\sqrt{-1}\right] \times \left[\sqrt{a}.\sqrt{-1}\right]$$
$$= \left(\sqrt{a}\right)^2.\left[\sqrt{-1}\right]^2 = a(-1) = -a \qquad \text{... (ii)}$$

Hence $\sqrt{-a}$ means the product of $\sqrt{a}$ and $\sqrt{-1}$. Thus a pure imaginary numbers can be expressed as the product of $\sqrt{-1}$ and a real numbers. Students must note carefully the results (i) and (ii). Keeping these results in view following computation is correct.

$$\sqrt{-a}.\sqrt{-b} = \sqrt{a}.\sqrt{-1}.\sqrt{b}.\sqrt{-1}$$
$$= \sqrt{a}.\sqrt{b}\left(\sqrt{-1}\right)^2 = -\sqrt{ab}$$

But the computation

$$\sqrt{-a}.\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab} \text{ is wrong.}$$

## Difference of two complex numbers

The difference of two complex numbers z = (x, y) and z' = (x', y') is defined by the equality

$$z - z' = z + (-z') = (x, y) + (-x', -y')$$
$$= (x + (-x'), y + (-y')) = (x - x', y - y')$$

## Division of two complex numbers

It is defined by the equality

$$z/z' = z(z')^{-1}, \text{provided}$$
$$z' \neq (0, 0)$$

we have $\frac{z}{z'} = (x, y)(x', y')^{-1}$

$$(x, y)\left(\frac{x'}{x'^2+y'^2}, -\frac{y'}{x'^2+y'^2}\right)$$
$$= \left(\frac{xx'}{x'^2+y'^2} + \frac{yy'}{x'^2+y'^2}, -\frac{xy'}{x'^2+y'^2} + \frac{yx'}{x'^2+y'^2}\right)$$
$$= \left(\frac{xx'+yy'}{x'^2+y'^2}, \frac{yx'-xy'}{x'^2+y'^2}\right)$$

provided $x'^2 + y'^2 \neq 0$

## Modulus and argument of a complex number

Let z = x + *i*y be any complex number.

If $x = r\cos\theta, y = r\sin\theta$, then $r = +\sqrt{x^2 + y^2}$ is called the modulus of the complex numbers z written as $|z|$ and $\theta = \tan^{-1}(y/x)$ is called the argument or amplitude of z written as arg. z.

It follows that |z| = 0 if and only if x = 0 and y = 0.

Geometrically, |z| is the distance of the point z from the origin.

It can be easily proved that

$$|z|^2 = |z^2|$$
$$\text{Re}\, z \leq |z| \text{and } Im\,(z) \leq |z|$$

Also argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi + \theta$, where $n = 0, \pm 1, \pm 2, \dots \dots$

The value of argument which satisfies the inequality

$$-\pi \leq \theta \leq \pi$$

is called the principal value of the argument. We remark that the argument of 0 is not defined.

**Remark:-**

It is evident from the definition of difference and modulus that $|z_1 - z_2|$ is the distance between the points $z_1$ and $z_2$. It follows that for fixed complex number $z_0$ and real number r, the equation $|z - z_0| = r$ represents a circle with centre $z_0$ and radius r.

## Polar form of a Complex Number

If r is the modulus and θ the argument of a complex numbers z, then $z = r(\cos\theta + i\sin\theta)$ is the polar form or trigonometrical form of z.

Since $e^{i\theta} = \cos\theta + i\sin\theta$, we can write $z = re^{i\theta}$. This is known as the exponential form of z.

∴ $x + iy = z = r(cos\theta + i\sin\theta) = re^{i\theta}$

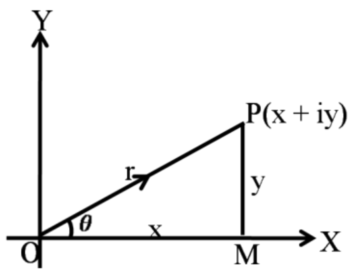
∴ $\tan\theta = \frac{y}{x} = \frac{Im\ z}{Re\ z}, \cos\theta = \frac{Re\ z}{|z|}, \sin\theta = \frac{Im\ z}{|z|}$

A complex numbers whose modulus is unity, i.e.

r = can be written as $z = e^{i\theta}$.

## The geometrical representation of complex numbers

We represent the complex numbers z = x + iy by a point P whose Cartesian co-ordinates are (x, y) referred to rectangular axes OX and OY, usually called real and imaginary axes respectively. Clearly the polar co-ordinates of P are (r, θ) where r is the modulus and θ the argument of complex number z. The plane whose points are represented by complex numbers is called **Argand plane** or **Argand diagram.** It is also called Complex plane or Gaussian plane.




**Note:-**

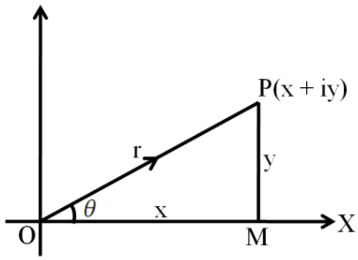
The complex number z is known as the affix of the point (x, y) which represents it.

## Vector representation of complex numbers

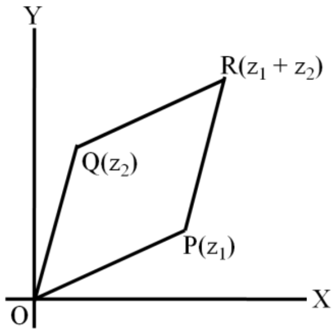
If P is the point (x, y) on the argand plane corresponding to the complex number z = x + iy referred to OX and OY as co-ordinate axes, the modulus and argument of z are represented by the magnitude and direction of the vector $\overrightarrow{OP}$ respectively and vice-versa.

The points on the argand plane representing the sum, and difference of two complex numbers.

**Sum:-** Let the complex numbers $z_1$ and $z_2$ be represented by the points P and Q on the argand plane.




Complete the parallelogram OPRQ. Then the mid-points of PQ and OR
Are the same. But mid-point of PQ is




$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

So that the co-ordinates of R are $(x_1 + x_2, y_1 + y_2)$. Thus the point R corresponds to the sum of the complex numbers $z_1$ and $z_2$.
In vector notation we have

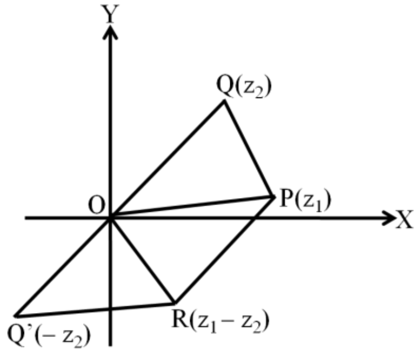
$$z_1 + z_2 = \overrightarrow{OP} + \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} \qquad \text{... (i)}$$

**Difference:-** We first represent $-z_2$ by Q' so that QQ' is bisected at O. Complete the parallelogram OPRQ''. Then the point R represents the complex number $z_1 - z_2$ since the mid-point of PQ' and OR are the same. As OQ is equal and parallel to RP, we see that ORPQ is a parallelogram, so that $\overrightarrow{OR} = \overrightarrow{QP}$.

Thus we have in vectorial notation

$$z_1 - z_2 = \overrightarrow{OP} - \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{QO}$$
$$= \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OR} = \overrightarrow{QP} \qquad \text{... (ii)}$$

It follows that the complex numbers $z_1 - z_2$ is represented by vector $\overrightarrow{QP}$, where the points P and Q represent the complex numbers $z_1$ and $z_2$ respectively.




## Conjugate to complex numbers

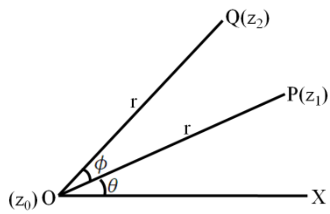
If z = x + iy, then the complex numbers x – iy is called the conjugate of the complex number z and is written as $\bar{z}$. It is easily seen that numbers conjugate to $z_1 + z_2$ and $z_1 z_2$ are $\overline{z_1} + \overline{z_2}$ and $\overline{z_1}\,\overline{z_2}$ respectively.

Also we have,

$$|z|^2 = z\bar{z}, 2x = 2Re(z) = z + \bar{z}$$

$$2iy = 2iIm(z) = z - \bar{z}.$$

It is clear that $|\bar{z}| = |z|, (\bar{\bar{z}}) = z$. Geometrically, the conjugate of z is the reflection (or image) of z in the real axis. If $(r, \theta)$ are polar co-ordinates of P, then the polar co-ordinates of its reflection P' are $(r, -\theta)$ so that we have amp $z = -$amp $\bar{z}$.




**Conjugate of $a + be^{i\theta}$:-**

Conjugate of $a + be^{i\theta}$ is not $a - be^{i\theta}$ but $a + be^{-i\theta}$ as shown below

$$a + be^{i\theta} = a + b(\cos\theta + i\sin\theta)$$

$$= (a + b\cos\theta) + ib\sin\theta$$

Its conjugate is $(a + b\cos\theta) - ib\sin\theta$

$$= a + b(\cos\theta - i\sin\theta) = a + be^{-i\theta}$$

**Note:-**

$$\bar{\imath} = -i, \bar{\imath}\ \bar{z} = -i\bar{z}.$$

***Illustration***

If $x + iy = \sqrt{\left(\frac{a+ib}{c+ib}\right)}$, prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$.

***Illustration***

(a): – If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.

(c): – If $x = \frac{-5+i\sqrt{3}}{2}$, prove that

$(x^2 + 5x)^2 + x(x + 5) = 42$

***Illustration***

If z = z + iy and $z^{1/3}$ = a – ib, then show that $\frac{x}{a}-\frac{y}{b}=4(a^2-b^2)$.

## The following relations be committed to memory

$i^4=1, i^{4n}=1, i^{4n+1}=i$

$i^{4n+2}=i^2=-1, i^{4n+3}=i^3=-i$

$(1+i)^2=1+i^2+2i=2i$

∴ $(1-i)^2=\text{conj.}=-2i, 1+i^2=0, 1-i^2=2$

$\frac{1+i}{1-i}=\frac{(1+i)^2}{1-i^2}=\frac{2i}{2}=i \quad \therefore \ \frac{1-i}{1+i}=\frac{1}{i}=-i.$

***Illustration***

Simplify the following:

(a): $\frac{3-i}{2+i}+\frac{3+i}{2-i}$ (b): $\frac{3}{1+i}-\frac{2}{2-i}+\frac{2}{1-i}$

***Solution***

(a):– $\frac{(3-i)(2-i)+(3+i)(2+i)}{4+1}$

$=\frac{1}{5}[(6-5i+i^2)+(6+5i+i^2)]=\frac{10}{5}=2.$

(b):– $\frac{3(1-i)}{2}-\frac{2(2+i)}{4+1}+\frac{2(1+i)}{2}$

$=\frac{1}{10}[15(1-i)-4(2+i)+10(1+i)]=\frac{1}{10}(17-9i).$

***Illustration***

The real part of (1 + i)2/(3 – i) is

(a) -1/5 (b) 1/3

(c) -1/3 (d) none of these.

***Solution***

$\frac{(1+i)^2}{3-i}=\frac{1+i^2+2i}{3-i}=\frac{2i(3+i)}{3^2-i^2}=\frac{6i-2}{10}$

∴ $Re\left(\frac{(1+i)^2}{3-i}\right)=-\frac{2}{10}=-\frac{1}{5}.$

***Illustration***

$\left(\frac{2i}{1+i}\right)^2$

(a) I (b) 2i

(c) 1 – I (d) 1 – 2i

***Solution***

The correct option is b.

$\frac{4i}{2i}=2i$